HAMILTONIAN STRUCTURE AND STABILITY OF RELATIVISTIC GRAVITATIONAL THEORIES

By

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The research contained within this thesis is concerned with the Hamiltonian structure and stability of several relativistic physical systems. Its primary focus involves a comparison of the Hamiltonian structure and stability properties of two alternative gravitational theories with General Relativity. Specifically, the Brans-Dicke and Kibble theories of gravity are the subject of study.

Working in the context of an Arnowitt-Deser-Misner splitting into space plus time, it may be demonstrated that the Vlasov-Einstein system, i.e., the collisionless Boltzmann equation of General Relativity, is Hamiltonian, and then this Hamiltonian character may be used to derive nontrivial criteria for linear and nonlinear stability of time-independent equilibria. Unlike all earlier work on the problem of stability, the formulation provided here is completely general,

incorporating no assumptions regarding spatial symmetries or the form of the equilibrium. The two alternative gravitational theories are studied by using the same methods.

The fundamental arena of physics is an infinite-dimensional phase space, coordinatized by the distribution function f, the spatial metric h_{ab}, and the conjugate momentum Π^{ab}. The Hamiltonian formulation entails the identification of a Lie bracket $\langle F, G \rangle$, defined for pairs of functionals $F[h_{ab}, \Pi^{ab}, f]$ and $G[h_{ab}, \Pi^{ab}, f]$, and a Hamiltonian function $H[h_{ab}, \Pi^{ab}, f]$, so chosen that the equations of motion $\partial_t F = \langle F, H \rangle$ for arbitrary F, with ∂_t a coordinate time derivative, are equivalent to the Vlasov-Einstein system. An analogous mathematical structure is deduced for the Vlasov-Brans-Dicke and Vlasov-Kibble Hamiltonian systems, where the gravitational field variables differ from General Relativity.

An explicit expression is derived for the most general dynamically accessible perturbation $\delta X = \{\delta f, \delta h_{ab}, \delta \Pi^{ab}\}$ which satisfies the Hamiltonian and momentum field constraints and the matter constraints associated with conservation of phase, and it is shown that all equilibria are energy extremals with respect to such δX, i.e., $\delta^{(1)} H[\delta X] = 0$. The sign of the second variation, $\delta^{(2)} H$, which is also computed, is thus related directly to the problem of linear stability. If $\delta^{(2)} H \geq 0$ for all dynamically accessible perturbations δX, the equilibrium is guaranteed to be linearly stable. The existence of some perturbation δX for which $\delta^{(2)} H[\delta X] \leq 0$ does not necessarily signal a linear instability.

CHAPTER 1
INTRODUCTION

Throughout this dissertation the Hamiltonian structure, and energetic stability, of several relativistic collisionless plasma systems will be studied. In chapter 2, the collisionless Vlasov-Newtonian cosmology and the fixed curved background spacetime collisionless Vlasov-Maxwell system will be examined. In the remaining three chapters, the Hamiltonian structure of three relativistic theories of gravitation will be analyzed in detail. Chapter 3 consists of a study of the Hamiltonian structure of the Vlasov-Einstein system (General Relativity), while chapter 4 is a corresponding study of the Vlasov-Brans-Dicke Hamiltonian system. Finally, chapter 5 deals with the Vlasov-Kibble Hamiltonian system.

The Hamiltonian of each gravitational theory will be written in terms of the ADM formalism (Arnowitt, Deser, and Misner (1962)). The use of this formalism facilitates a convenient 3+1 split into space and time variables. From a dynamical standpoint, this makes the analysis of static equilibrium matter configurations particularly straightforward. For each theory, there are field variables and corresponding conjugate momenta. Following Dirac (1950), one can construct a bracket out of these canonically conjugate dynamical variable pairs. The equations of dynamics for the conjugate variables may be derived from the bracket. This applies to the field variables of the gravitation theories.

A Boltzmann distribution f will be used to describe the matter distribution. This distribution corresponds to the case of a collisionless matter distribution and it will satisfy the Liouville, i.e., collisionless Boltzmann equation. This variable f does not possess a conjugate variable. Therefore, it is not straightforward to construct a Dirac bracket for it.

1.1 Theory of Casimir Invariants

Early work with the theory of Casimir invariants dates back to the late 1950's with the research of W. Newcomb (cf. the appendix in Bernstein (1958)). Newcomb considered a perturbation of a plasma Hamiltonian for which the entropy, serving as a Casimir invariant, remains fixed. His idea, which went as follows, was developed to show that general small motions of a collisionless plasma about thermal equilibrium cannot exhibit exponential growth in time. Newcomb utilized the fact that entropy is a constant of the motion for collisionless systems.

We will let

$$f_o(v) = N\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[\frac{-mv^2}{2k_B T}\right] \tag{1.1}$$

denote the Maxwell distribution. In equation (1.1) the particle density is denoted by N and Boltzmann's constant is denoted by k_B. The mass and velocity of the plasma particles are denoted, respectively, as m and v. The temperature at thermal equilibrium is denoted by T. Newcomb worked with the following expression for the entropy

$$S = -k_B \sum \int d^3r d^3v [f \ln f - f_o \ln f_o], \tag{1.2}$$

where the summation is taken over all species of particle present in the plasma. Since $S = 0$ for thermal equilibrium, the expression for entropy (1.2) is normalized.

For a departure from thermal equilibrium, that is spatially bounded, the integral (1.2) is well defined. For spatial disturbances of the form $e^{+i\vec{k}\cdot\vec{r}}$, the integration may be taken over a cube of side length $2\pi/k$ in which one face is perpendicular to the wave vector $\vec{k}$. One may compute the time rate of change of S to be

$$\frac{dS}{dt} = -k_B \sum \int d^3r d^3v \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial t} \ln f\right]. \tag{1.3}$$

The term $\partial f/\partial t$ will vanish due to the conservation of the total number of particles (factor the partial time derivative out of the integral). For the rest of the integral, it is necessary to utilize the Boltzmann equation (Chapman and Cowling (1939))

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \frac{q}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}} = \frac{\partial f}{\partial t}_{coll.}, \tag{1.4}$$

where $\partial f/\partial t_{coll.}$ represents the rate of change of the distribution function arising from particle collisions. For a collisionless plasma, one may set

$$\partial f/\partial t_{coll.} = 0.$$

Substitution of this collisionless Boltzmann, i.e., Liouville equation into equation (1.3) yields the result

$$\frac{dS}{dt} = k_B \sum \int d^3r d^3v \ln f \left\{ \frac{\partial}{\partial \vec{r}}\cdot(\vec{v}f) + \frac{\partial}{\partial \vec{v}}\cdot\left[\frac{q}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)f\right]\right\}. \tag{1.5}$$

Integration by parts leads to the following modified version of equation (1.5)

$$\frac{dS}{dt} = -k_B \sum \int d^3r d^3v \left[\vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \frac{q}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}}\right] = 0. \tag{1.6}$$

Consequently, the entropy is a constant of the motion known as a Casimir invariant.

The total energy of the plasma system is given by

$$W = \frac{1}{8\pi}\int d^3r(\vec{E}^2 + \vec{B}^2 - \vec{B}_o^2) + \sum \int d^3r d^3v \frac{m}{2}v^2(f - f_o), \tag{1.7}$$

where $\vec{B}_o$ denotes a constant external magnetic field. The energy is normalized since $W = 0$ at equilibrium. The Maxwell equations are

$$\nabla\cdot\vec{E} = 4\pi\rho, \quad c\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$\nabla\cdot\vec{B} = 0, \quad c\nabla\times\vec{B} = 4\pi\vec{J} + \frac{\partial \vec{E}}{\partial t}, \tag{1.8}$$

where the charge density ρ and current density $\vec{J}$ are given by

$$\rho = \sum Ze \int d^3vf, \quad \vec{J} = \sum Ze \int d^3v\vec{v}f.$$

Using these equations and the Boltzmann equation (1.4) allows one to prove that W is a constant of the motion. Therefore, the combined system $F = W + S$ is a constant of the motion.

One may expand the distribution function in terms of powers of a small initial departure from thermal equilibrium

$$f = f_o + f_1 + f_2 + ... \quad . \tag{1.9}$$

There are corresponding changes in the electric and magnetic fields

$$\vec{E} = 0 + \vec{E}_1 + \vec{E}_2 + ...,$$

$$\vec{B} = \vec{B}_o + \vec{B}_1 + \vec{B}_2 + ... \quad . \tag{1.10}$$

To second order the entropy has the following form

$$S = -k_B \sum \int d^3rd^3v \Big\{ f_1\Big(1 + A - \frac{mv^2}{2k_BT}\Big) + f_2\Big(1 + A - \frac{mv^2}{2k_BT}\Big) + \frac{{f_1}^2}{2f_o} \Big\}, \tag{1.11}$$

where the constant $A = \frac{3}{2}\ln[N^{2/3}m/2\pi k_BT]$. Due to particle conservation, the terms above in $(1 + A)$ will vanish. Expressing equation (1.7) to second order allows us to write equation (1.11) as

$$S = -k_B \Big\{ \Big(\frac{1}{8\pi k_BT}\Big) \int d^3r[\vec{E}_1^2 + \vec{B}_1^2] + \frac{1}{2} \int d^3rd^3v\frac{{f_1}^2}{f_o} \Big\}. \tag{1.12}$$

We obtain a bracketed expression which is essentially a sum of positive terms. However, we know from equation (1.6) that S is a constant of the motion. Consequently, it is not possible for any of the quantities $\vec{E}_1$, $\vec{B}_1$ or f_1 to increase monotonically. For example, an exponential increase in time of f_1 would not be permitted.

This argument by Newcomb disproved the purported proof by Gordeyev (1952) that such an exponential growth of the distribution f could occur for a collisionless plasma in thermal equilibrium. Newcomb considered the conservation of only one constraint. However, in later work, Kruskal and Oberman (1958) considered the conservation of a general set of Casimir constraints in a paper dealing with plasma stability. Gardner (1963) provided additional insight with the recognition of the importance of phase space conservation in regard to electrostatic Vlasov stability theory. His arguments were as follows.

One may assume that $f(t, \vec{x}, \vec{v})$ denotes the distribution function of one species of particle present in a plasma. Since the plasma is collisionless, the distribution will satisfy the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f + \frac{\vec{F}}{m} \cdot \nabla_{\vec{v}} f = 0, \tag{1.13}$$

where the force $\vec{F}(t, \vec{x}, \vec{v})$ obeys the condition

$$\nabla_{\vec{v}} \cdot \vec{F} = 0. \tag{1.14}$$

The motion of one particle is dictated by the usual dynamical equations

$$d\vec{x}/dt = \vec{v}, \tag{1.15}$$

$$m d\vec{v}/dt = \vec{F}. \tag{1.16}$$

Equations (1.15) and (1.16) define a flow in $\vec{x}, \vec{v}$ phase space. By combining equations (1.14)-(1.16) we see that the flow is incompressible. Also, equation (1.13) states that the value of f, at the phase space position of any fixed particle, will not change as the particle moves about in phase space. In other words, the distribution remains invariant when translated along any phase space trajectory that is consistent with the dynamical equations. One may interpret the phase space motion as that of an incompressible fluid flow of variable density

f. The initial value of f will be

$$f(0, \vec{x}, \vec{v}) = f_o(\vec{x}, \vec{v}). \tag{1.17}$$

The kinetic energy of the plasma will take the form

$$W(t) = \frac{1}{2} m \int \int v^2 f(t, \vec{x}, \vec{v}) d\vec{x} d\vec{v}. \tag{1.18}$$

One may integrate in $\vec{x}$ over a finite box if the assumption of periodic $\vec{x}$ dependence of the distribution f is made.

With the initial value of f_o, we wish to determine a lower bound for the kinetic energy of the plasma. Such a lower bound may be written as

$$W_1 = \frac{1}{2} m \int \int v^2 f_1(\vec{v}) d\vec{x} d\vec{v}, \tag{1.19}$$

where $f_1(\vec{v})$, corresponding to a later value of the evolving distribution, represents a monotone decreasing function of v^2. Also, for any number $\alpha > 0$, the phase space volume of the region where $f_1(\vec{v}) > \alpha$ is equal to the phase space volume of the region where $f_o(\vec{x}, \vec{v}) > \alpha$. This condition holds because the phase space flow is incompressible with the value of f at a particle conserved during subsequent evolution of the system.

With these enforced conditions, one wishes to determine what the minimum possible value for $W(t)$ will be. It is straightforward to see that the lowest-energy state corresponds to the case in which more massive particles (f large) are even closer to $\vec{v} = 0$. For example, if we choose two particles f and f' with respective velocities v and v', then, if we are dealing with the lowest-energy state, the condition $f > f'$ implies that $v < v'$. This state corresponds to $f_1(\vec{v})$. Consequently, the amount of kinetic energy originally present in $f_o(\vec{x}, \vec{v})$, which can be given up to the field, is at most

$$W(0) - W_1. \tag{1.20}$$

Afterwards Lynden-Bell and Sanitt (1969) applied the method of conserved constraints to study the energy associated with a linearized perturbation of a galactic dynamical system. They referred to this as "a trick due to Newcomb." However, it was left to Bartholomew (1971) to finally deduce the mathematical form of the phase-preserving perturbation of the Boltzmann distribution which leaves general Casimir constraints invariant. Bartholomew's paper dealt with the problem of galactic stability.

Bartholomew treated a galaxy as a system of point masses moving under the influence of the collective gravitational potential of the entire system. One may define a phase space function $F = F(q_i, p_i, t)$ over the space of generalized coordinates q_i and conjugate momenta p_i. A phase space volume element may be defined as $d\tau = dq_1 dq_2 dq_3 dp_1 dp_2 dp_3$. The continuity of flow through volume $d\tau$ of phase space may be expressed as

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial q_i}\left(F\frac{dq_i}{dt}\right) + \frac{\partial}{\partial p_i}\left(F\frac{dp_i}{dt}\right) = 0. \tag{1.21}$$

Using Hamilton's canonical equations of motion (for the case of a unit mass) allows one to rewrite equation (1.21) as

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial q_i}\left(F\frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(F\frac{\partial H}{\partial q_i}\right) = 0. \tag{1.22}$$

The Poisson bracket may be defined as

$$[A, B] = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}. \tag{1.23}$$

This enables one to redefine equation (1.22) as

$$\frac{\partial F}{\partial t} + [F, H] = 0. \tag{1.24}$$

The Hamiltonian H may be written as the sum of two terms. The first term, which corresponds to the kinetic energy, is a function of the generalized coordinates. The second

term corresponds to the gravitational potential Ψ. Using Cartesian coordinates one may write the matter density as

$$\rho = \int F d^3 p, \tag{1.25}$$

which yields the gravitational term

$$\Psi = G \int \frac{\rho' d^3 r'}{|\vec{r} - \vec{r}'|} = G \int \frac{F' d\tau'}{|\vec{r} - \vec{r}'|}. \tag{1.26}$$

For the sake of convenience, one may give the gravitational term of the Hamiltonian a more general formulation

$$-\int K(q_i, p_i; {q'}_i, {p'}_i) F({q'}_i, {p'}_i) d\tau', \tag{1.27}$$

where K is a symmetric kernel. The second gravitational term has a linear dependence on the distribution F. One may rewrite equation (1.22) as

$$\frac{\partial F}{\partial t} + \frac{\partial q_i}{\partial t}\frac{\partial F}{\partial q_i} + \frac{\partial p_i}{\partial t}\frac{\partial F}{\partial p_i} = 0, \tag{1.28}$$

which corresponds to

$$\frac{dF(q_i, p_i, t)}{dt} = 0. \tag{1.29}$$

Assuming that the galactic system is perturbed from equilibrium by an infinitesimal amount

$$F + \delta F \tag{1.30}$$

leads to a corresponding change in the Hamiltonian

$$\delta H = -\int K \delta F' d\tau'. \tag{1.31}$$

The Vlasov equation (1.31) will undergo a similar linearized perturbation

$$\frac{\partial}{\partial t}(\delta F) + [\delta F, H] + [F, \delta H] = 0. \tag{1.32}$$

One wishes to determine whether, from the perspective of stability theory, the perturbation δF will experience monotonic growth or will oscillate about equilibrium. As discussed in Bartholomew (1971) the relevant perturbations δF are those which conserve angular momentum and total mass, and are generated by a canonical transformation. We insist on a canonical transformation because the Hamiltonian equations

$$\frac{dq_i}{dt} = [q_i, H] \quad \frac{dp_i}{dt} = [p_i, H] \tag{1.33}$$

generate translations in phase space

$$q_i + dq_i,$$
$$p_i + dp_i, \tag{1.34}$$

by means of an infinitesimal canonical transformation. We are only interested in perturbations which are consistent with the equations of dynamics.

One wishes to move stars from the phase space point (q_i, p_i) to an infinitesimally close nearby point $(q_i + \xi_i, p_i + \eta_i)$. The perturbation corresponds to an infinitesimal canonical transformation provided that there exists a generating function g such that

$$\xi_i = \frac{\partial g}{\partial p_i}, \tag{1.35a}$$

$$\eta_i = -\frac{\partial g}{\partial q_i}. \tag{1.35b}$$

A more compact notation for equations (1.35) may be introduced by use of the Poisson bracket

$$\xi_i = [q_i, g], \quad \eta_i = [p_i, g]. \tag{1.36}$$

The change in F may be found by writing the following equation

$$\delta F + \frac{\partial}{\partial q_i}(F\xi_i) + \frac{\partial}{\partial p_i}(F\eta_i) = 0, \tag{1.37}$$

which follows by analogy with equation (1.21). This leads to the proper form for the phase-preserving perturbation

$$\delta F = -[F, g], \tag{1.38}$$

where δF is defined in terms of the generating function g. Substitution of equation (1.38) into equation (1.32) yields the result

$$\left[F, \frac{\partial g}{\partial t}\right] = -[[F, g], H] + [F, \delta H]. \tag{1.39}$$

Multiplication of this equation by $\partial g/\partial t$ and integration over all phase space leads to

$$\int \frac{\partial g}{\partial t}\left[F, \frac{\partial g}{\partial t}\right] d\tau = -\int \left[F, \frac{\partial g}{\partial t}\right][H, g] d\tau - \int \int \left[F, \frac{\partial g}{\partial t}\right] K[F', g'] d\tau' d\tau. \tag{1.40}$$

The left-hand side of this equation vanishes by virtue of antisymmetry

$$\int \frac{\partial g_1}{\partial t}\left[F, \frac{\partial g_2}{\partial t}\right] d\tau = -\int \frac{\partial g_2}{\partial t}\left[F, \frac{\partial g_1}{\partial t}\right] d\tau. \tag{1.41}$$

The right-hand side of equation (1.40) can be proven symmetric in g and $\partial g/\partial t$, allowing one to rewrite it as

$$\frac{d}{dt}\left\{-\frac{1}{2}\int [F, g][H, g] d\tau - \frac{1}{2}\int \int [F, g] K[F', g'] d\tau d\tau'\right\} = 0. \tag{1.42}$$

The functional in brackets corresponds to a second order energy perturbation.

Assuming a stationary reference frame, the Hamiltonian H is the energy per unit mass. The total energy E, for a fixed potential field, would be given by

$$E = \int F H d\tau. \tag{1.43}$$

However, if a perturbation of F alters the energy of other material by varying the potential, then the equation (1.43) can be written in the form

$$E = \int F H d\tau + \frac{1}{2}\int \int F K F' d\tau d\tau'. \tag{1.44}$$

A first order perturbation of this energy can be calculated to be

$$\delta E = \int (F+\delta F)(H+\delta H)d\tau + \frac{1}{2}\int\int (F+\delta F)K(F'+\delta F')d\tau d\tau' - E \qquad (1.45)$$

$$= \int H\delta F d\tau - \frac{1}{2}\int\int \delta F K \delta F' d\tau d\tau'.$$

Therefore, the first order change in energy is

$$\delta E = -\int H[F,g]d\tau = \int [F,H]g d\tau = 0. \qquad (1.46)$$

Thus, there is no change in the energy of an equilibrium for a lowest order perturbation.

1.2 Constrained Hamiltonian Dynamics

It was Dirac (1950) who first studied constrained Hamiltonian dynamics. We may follow his treatment of the subject. Typically, one may consider a dynamical system of N degrees of freedom described by generalized coordinates q_n (where n takes the values 1, 2,..., N) and velocities $dq_n/dt = \dot{q}_n$. The dynamics of the system is described by a Lagrangian

$$L \equiv L(q_n, \dot{q}_n), \qquad (1.47)$$

which possesses the corresponding momenta

$$p_n = \frac{\partial L}{\partial \dot{q}_n}. \qquad (1.48)$$

The variables q_n, $\dot{q}_n$, and p_n may be varied by a small amount δq_n, $\delta \dot{q}_n$, and δp_n of the order of ϵ. In our calculations, we only work to the order of accuracy of ϵ. However, variation of equation (1.48) leads to difficulties because the left-hand side will differ from the right-hand side by a quantity of order ϵ. Following Dirac (1950), we may class equations as to whether they remain accurate to order ϵ under variation or not. The Lagrangian (1.47) remains valid under variation, since, by definition, the variation in L must equal the variation of $L(q_n, \dot{q}_n)$. Equations of this type are referred to as strong equations, while equations, such

as (1.48), are referred to as weak equations. The following algebraic rules exist for the weak and strong equations; for general cases

$$\delta A = 0 \tag{1.49}$$

($A \equiv 0$ is a strong equation) and

$$\delta X \neq 0 \tag{1.50}$$

($X = 0$ is a weak equation), where the notations $\equiv$ for strong equivalence and $=$ for weak equivalence have been employed. We can conclude that

$$\delta X^2 = 2X\delta X = 0$$

where $X = 0$ is a weak equation. Consequently, we have deduced the strong equation $X^2 \equiv 0$. A strong equation

$$X_1 X_2 \equiv 0,$$

can be constructed from two weak equations $X_1 = 0$ and $X_2 = 0$.

The standard case occurs when the N quantities $\partial L/\partial \dot{q}_n$ of equation (1.48) are all independent functions of the N velocities $\dot{q}_n$. For this case one may determine each velocity $\dot{q}_n$ as a function of the coordinates q_n and momenta p_n. However, if the $\partial L/\partial \dot{q}_n$ are not independent functions of the velocities $\dot{q}_n$, then we may eliminate $\dot{q}$'s from equation (1.48) and derive one or more equations

$$\phi(q, p) = 0 \tag{1.51}$$

which are functions of coordinates q and momenta p. The equations (1.51) are weak equations. One may consider a complete set of independent equations (1.51)

$$\phi_m(q, p) = 0, \quad m = 1, 2, ..., M. \tag{1.52}$$

A function of the q's and p's, which vanishes due to equation (1.48), may be written as a linear function of the ϕ_m with coefficients that are functions of the q's and p's. The strong and weak equations may be given a geometrical interpretation. Initially, one may visualize a 3N dimensional space with coordinates $q_n, \dot{q}_n$, and p_n. This space will possess a 2N dimensional hypersurface where equation (1.48) holds. This hypersurface will be called $\mathcal{R}$. Equations (1.51), which are deduced from equation (1.48), will also hold on this hypersurface. We may also define a 3N dimensional region of all points that are within a distance of order ϵ from the hypersurface $\mathcal{R}$. This region will be referred to as $\mathcal{R}_\epsilon$. A weak equation will hold in the region $\mathcal{R}$, while a strong equation will hold in the region $\mathcal{R}_\epsilon$. The Hamiltonian is defined in the usual manner as

$$H = p_n \dot{q}_n - L, \tag{1.53}$$

where the standard summation convention is employed. The Hamiltonian (1.53) can be subjected to a variation as follows

$$\begin{aligned}\delta H &= \delta(p_n \dot{q}_n - L) \\ &= p_n \delta\dot{q}_n + \dot{q}_n \delta p_n - \partial L/\partial q_n \delta q_n - \partial L/\partial \dot{q}_n \delta \dot{q}_n \\ &= \dot{q}_n \delta p_n - \partial L/\partial q_n \delta q_n.\end{aligned} \tag{1.54}$$

Consequently, δH does not depend on the $\delta\dot{q}$'s. This result holds irregardless of whether or not the standard case applies. The definition (equation 1.53) of the Hamiltonian holds throughout the 3N dimensional space of q's, $\dot{q}$'s and p's. The result (equation 1.54) holds in the region $\mathcal{R}_\epsilon$ to first order. Therefore, assuming that q and p remain constant, a first order variation in the velocities $\dot{q}_n$ corresponds to a second order variation in H. Thus, for constant q_n and p_n, the variation in H, corresponding to a finite variation in $\dot{q}_n$, will be of

the first order provided that the variation is performed in $\mathcal{R}_\epsilon$. The variation in H will be zero if it is performed in the region $\mathcal{R}$. It follows that, in the region $\mathcal{R}$, the Hamiltonian is a function of q and p. For the region $\mathcal{R}$ one may define the weak Hamiltonian function

$$H = h(q,p). \tag{1.55}$$

The function h is the ordinary Hamiltonian in the standard case. The following general variation

$$\delta(H-h) = \left(\dot{q}_n - \frac{\partial h}{\partial p_n}\right)\delta p_n - \left(\frac{\partial L}{\partial q_n} + \frac{\partial h}{\partial q_n}\right)\delta q_n \tag{1.56}$$

may be performed from a point in $\mathcal{R}$. If the variation is within the region $\mathcal{R}$, then it is obvious that $\delta(H-h) = 0$. For such a variation of δq_n and δp_n, the dynamical equations (1.48) are preserved by an appropriate choice of $\delta\dot{q}_n$. The δq's and δp's are restricted in the sense that $\delta\phi_m = 0$ for all values of m. However, for arbitrary perturbations δq, δp which do not satisfy this condition, we compute

$$\delta(H-h) = v_m \delta\phi_m \tag{1.57}$$

for a suitable choice of coefficients. These coefficients v_m are functions of the q's and p's.

Equation (1.57) can be reformulated as

$$\delta(H - h - v_m\phi_m) = \delta(H-h) - v_m\delta\phi_m - \phi_m\delta v_m = 0, \tag{1.58}$$

where equation (1.52) has been used. One may use equation (1.58) to deduce the following strong equation

$$H \equiv h + v_m\phi_m \tag{1.59}$$

which holds to the first order in $\mathcal{R}_\epsilon$. A variation of equation (1.59) yields the result

$$\delta H = \delta h + v_m\delta\phi_m$$

$$= \frac{\partial h}{\partial p_n}\delta p_n + \frac{\partial h}{\partial q_n}\delta q_n + v_m\left(\frac{\partial \phi_m}{\partial p_n}\delta p_n + \frac{\partial \phi_m}{\partial q_n}\delta q_n\right). \tag{1.60}$$

Comparison of equation (1.60) with equation (1.54) yields the following expressions

$$\dot{q}_n = \frac{\partial h}{\partial p_n} + v_m \frac{\partial \phi_m}{\partial p_n}, \tag{1.61a}$$

$$-\frac{\partial L}{\partial q_n} = \frac{\partial h}{\partial q_n} + v_m \frac{\partial \phi_m}{\partial q_n}. \tag{1.61b}$$

The usual Lagrangian equations of motion

$$\dot{p}_n = \frac{\partial L}{\partial q_n}$$

can be combined with equation (1.61b) to yield the Hamiltonian equation of motion

$$\dot{p}_n = -\frac{\partial h}{\partial q_n} - v_m \frac{\partial \phi_m}{\partial q_n}. \tag{1.62}$$

The Hamiltonian equation corresponding to equation (1.62) is equation (1.61a).

It is convenient to formulate the Hamiltonian equations of motion in terms of the Poisson bracket notation. Any two functions ξ and η of the q's and p's possess the Poisson bracket

$$[\xi, \eta] = \frac{\partial \xi}{\partial q_n}\frac{\partial \eta}{\partial p_n} - \frac{\partial \xi}{\partial p_n}\frac{\partial \eta}{\partial q_n}. \tag{1.63}$$

Poisson brackets are, by their definition, subject to the following rules

$$[\xi, \eta] = -[\eta, \xi], \tag{1.64a}$$

$$[\xi, f(\eta_1, \eta_2, ...)] = \frac{\partial f}{\partial \eta_1}[\xi, \eta_1] + \frac{\partial f}{\partial \eta_2}[\xi, \eta_2] + ..., \tag{1.64b}$$

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0. \tag{1.64c}$$

According to Dirac (1950) the notion of a Poisson bracket may be extended to include functions of $\dot{q}$'s where the velocities are not expressible as functions of q's and p's. These

generalized Poisson brackets are still subject to the properties (1.64a)-(1.64c). The following weak equations

$$\frac{\partial A}{\partial q_n} = 0, \quad \frac{\partial A}{\partial \dot{q}_n} = 0, \quad \frac{\partial A}{\partial p_n} = 0 \tag{1.65}$$

can be derived from the strong equation $A \equiv 0$. For any ξ one may deduce that

$$[\xi, A] = 0 \tag{1.66}$$

by using equation (1.64b). It is possible, although by no means necessary, that $[\xi, A] \equiv 0$. However, for a weak equation $X = 0$ one cannot generally infer that $[\xi, X] = 0$.

If g is a function of the q's and p's then the Hamiltonian equations (1.61a) and (1.62) allow us to write

$$\begin{aligned} \dot{g} &= \frac{\partial g}{\partial q_n}\left(\frac{\partial h}{\partial p_n} + v_m \frac{\partial \phi_m}{\partial p_n}\right) - \frac{\partial g}{\partial p_n}\left(\frac{\partial h}{\partial q_n} + v_m \frac{\partial \phi_m}{\partial q_n}\right) \\ &= [g, h] + v_m[g, \phi_m]. \end{aligned} \tag{1.67}$$

Using equation (1.52) allows one to rewrite this expression as

$$\dot{g} = [g, h] + v_m[g, \phi_m] + [g, v_m]\phi_m = [g, H], \tag{1.68}$$

which is the generalized Hamiltonian equation of motion.

If the Lagrangian is homogeneous of the first degree in the velocities $\dot{q}_n$, then the momenta (1.48) will be homogeneous of degree zero in the $\dot{q}$'s. Consequently, they will depend only on the ratios of the $\dot{q}$'s. It follows that the p's cannot all be independent of the $\dot{q}$'s because there are only N - 1 independent ratios of the $\dot{q}$'s corresponding to N p's. Therefore, at least one relation of type (1.51) must exist between the q's and the p's. The Euler theorem is applicable in this case and leads to

$$L = \dot{q}_n \frac{\partial L}{\partial \dot{q}_n},$$

$$L = \dot{q}_n p_n. \tag{1.69}$$

This yields the following weak equation

$$H = 0 \tag{1.70}$$

which holds in the region $\mathcal{R}$. Therefore, we can assume that $h = 0$ and that

$$H = v_m \phi_m. \tag{1.71}$$

It follows that the general equation of motion reduces to the form

$$\dot{g} = v_m [g, \phi_m]. \tag{1.72}$$

As noted by Dirac (1950), the Lagrangian for any dynamical system can be made to satisfy the condition for homogeneous velocities by using an additional coordinate q_o to represent the time t and the condition $\dot{q}_o = 1$ is then used to make the Lagrangian homogeneous of the first degree in all the velocities, including $\dot{q}_o$. Therefore, without any loss of generality, one may consider only the homogeneous velocity theory.

The equations of motion must maintain the validity of the constraint equations (1.52). It follows that substitution of $\phi_{m'}$ for g in equation (1.72) must yield the set of equations

$$v_m [\phi_m, \phi_{m'}] = 0. \tag{1.73}$$

The system of equations (1.73) will be assumed to be reduced as much as possible with the set of equations (1.52). Each of the resulting equations must fall into one of the four categories:

Type 1. It involves some of the variables v_m.

Type 2. It involves the variables q and p and takes the form $\chi(q, p) = 0$.

However, it is independent of the variables v_m and independent of the

equations ϕ_m (1.52).

Type 3. It reduces to the form $0 = 0$.

Type 4. It reduces to the form $1 = 0$. An equation of type 2 will lead us to another consistency condition that is analogous to equation (1.73), namely

$$v_m[\phi_m, \chi] = 0, \tag{1.74}$$

for the constraint

$$\chi(q, p) = 0. \tag{1.75a}$$

The new consistency condition (1.74) may be reduced as far as possible by using eqs. (1.52) and (1.75a). Once again, eq. (1.74) will be one of the four types. If it is type 2, then we will obtain another consistency condition. Therefore, we must repeat the process until an equation of another type is obtained. If we obtain an equation of type 4, then the equations of dynamics are inconsistent. Therefore, we may ignore this case. Equations of type 3 are identically satisfied. It is necessary to examine the cases of type 1 and type 2.

A complete set of equations of type 2 may be written

$$\chi_k(q, p) = 0, \quad k = 1, 2, ..., K. \tag{1.75b}$$

The equations (1.75b) are chosen to exhibit order ϵ variations such as the constraint equations (1.52); both sets are weak. We define a constraint $\phi_{m'}$ to be a first class constraint if its Poisson bracket with all other constraints ϕ and χ vanishes, i.e., if

$$[\phi_{m'}, \phi_m] = 0, \quad m = 1, 2, ..., M$$

$$[\phi_{m'}, \chi_k] = 0, \quad k = 1, 2, ..., K. \tag{1.76}$$

Equations (1.76) only have to hold as consequences of $\phi_m = 0$ and $\chi_k = 0$, i.e., they can be weak. A constraint ϕ_m not satisfying these conditions is termed second class.

A linear transformation may be performed on the constraints ϕ_m

$$\phi_m{}^* = \gamma_{mm'}\phi_{m'}. \tag{1.77}$$

The γ matrix coefficients are functions of the coordinates q and momenta p such that their determinant will not vanish in the weak sense. It follows that the ϕ^* constraints are completely equivalent to the ϕ constraints. A transformation of this kind may be performed to bring as many constraints as possible into the first class. The first class constraints will be denoted ϕ_α and the second class constraints will be denoted ϕ_β, where $\beta = 1, 2, ..., B$ and $\alpha = B+1, B+2, ..., M$. The consistency conditions will reduce to the format

$$v_\beta[\phi_\beta, \phi_{\beta'}] = 0, \quad \beta, \beta' = 1, 2, ..., B$$

$$v_\beta[\phi_\beta, \chi_k] = 0, \quad k = 1, 2, ..., K. \tag{1.78}$$

These are all of the type 1 equations. These equations prove that either the v_β's must all vanish or that the matrix is, in the weak sense, of rank less than B. Dirac (1950) proved that the first possibility is correct. Therefore, if the maximum number of ϕ's possible have been placed in the first class, the v's corresponding to the second class constraints will all be zero. Therefore, the Hamiltonian H (1.71) can be written in terms of only the first class constraints as

$$H = v_\alpha \phi_\alpha \tag{1.79}$$

and the general equation of motion becomes

$$\dot{g} = v_\alpha[g, \phi_\alpha]. \tag{1.80}$$

We may form a determinant from a set of functions θ_s (s = 1, 2, ... , S) of the q's and p's such that the determinant is nonvanishing in the weak sense. Let $c_{ss'}$ correspond to the

cofactor of $[\theta_s, \theta_{s'}]$ divided by the determinant Δ. It follows that

$$c_{ss'} \equiv -c_{s's} \tag{1.81}$$

and

$$c_{ss'}[\theta_s, \theta_{s''}] \equiv \delta_{s's''}. \tag{1.82}$$

Dirac (1950) defined a new Poisson bracket $[\xi, \eta]^*$ for any two quantities ξ and η as

$$[\xi, \eta]^* \equiv [\xi, \eta] + [\xi, \theta_s]c_{ss'}[\theta_{s'}, \eta]. \tag{1.83}$$

The new Poisson brackets satisfy the three conditions on Poisson brackets (1.64a)-(1.64c). For any ξ we deduce the following condition on the new Poisson brackets

$$\begin{aligned}[\xi, \theta_s]^* &\equiv [\xi, \theta_s] + [\xi, \theta_{s'}]c_{s's''}[\theta_{s''}, \theta_s] \\ &= [\xi, \theta_s] - [\xi, \theta_{s'}]\delta_{s's} \\ &= 0.\end{aligned} \tag{1.84}$$

We may choose the θ's to consist entirely of ϕ's and χ's. The determinant Δ will vanish unless the ϕ's are second class. It follows that $[\theta_s, H] = 0$ for every value of s. Therefore, the new Poisson brackets give the Hamiltonian equations of motion

$$[g, H]^* = [g, H] = \dot{g}, \tag{1.85}$$

where g is any function of the q's and p's.

1.3 Covariant Poisson Brackets for Relativistic Field Theories

J.E. Marsden, R. Montgomery, P.J. Morrison, and W.B. Thompson (1986) were the first to write the equations of some specific relativistic field theories in covariant Poisson bracket form. They demonstrated that the field equations are equivalent to the following bracket

$$\{F, S\} = 0. \tag{1.86}$$

In this expression F is an arbitrary function of the fields and S is an action integral. The relativistic fields were grouped into two categories: (i) pure fields and (ii) media fields. The pure fields, corresponding typically to gauge fields, will be associated with functionals F of the basic field variables ϕ^A and their conjugate momenta $\Pi_A{}^\mu$. The media fields, which describe plasmas and relativistic fluids, are described by functionals F of only the field variables (without the conjugate momenta). An example of a media field would be a Boltzmann distribution f which possesses no corresponding conjugate momenta. It was noted in Marsden et al. (1986) that equation (1.86) satisfies Jacobi's Identity and the standard properties of Poisson brackets. Consequently, the space of fields is a Poisson Manifold (cf. Dirac (1950)).

For pure fields, the covariant bracket possesses a space-time vector field V^μ. If a 3+1 Dirac-ADM decomposition is implemented, then this vector field V^μ corresponds to a choice of foliation of the space-time manifold. In the case of media fields, the bracket is a covariant extension of the Lie-Poisson type discussed in section 1.2.

Marsden et al. (1986) deduced their results for relativistic field theories by first considering the case of elementary particle mechanics. The familiar canonical Hamiltonian equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{1.87a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \tag{1.87b}$$

follow from an initial variation $\delta S = 0$ of the action integral

$$S[\Gamma] = \int (p_i \dot{q}^i - H(q,p))dt. \tag{1.88}$$

The integral is treated as a functional on Γ, the phase space of trajectories $(q(t), p(t))$ associated with movement of a particle. Appropriate boundary conditions (cf. Arnold (1978,

p. 243)) are imposed on this phase space. The variational principle may be formulated in terms of a Poisson bracket defined on Γ. For phase space functionals F and G, one may define the bracket

$$\{F,G\}(\Gamma) = \int \left(\frac{\delta F}{\delta q^i}\frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i}\frac{\delta F}{\delta p_i} \right) dt, \tag{1.89}$$

where the functional derivatives take the form

$$\frac{d}{ds}F(\Gamma + s\delta\Gamma)|_{s=0} = \int \left(\frac{\delta F}{\delta \Gamma}\delta\Gamma \right) dt = \int \left(\frac{\delta F}{\delta q^i}\delta q^i + \frac{\delta F}{\delta p_i}\delta p_i \right) dt \tag{1.90}$$

and the variations $\delta\Gamma$ vanish at the endpoints of Γ. One may verify that Γ solves Hamilton's equation iff

$$\{F,S\}(\Gamma) = 0, \tag{1.91}$$

for arbitrary functionals F. Marsden et al. (1986) generalized this bracket operation to relativistic field theories. This covariant theory, however, does not single out a single space-time direction, but treats space and time in an equivalent manner.

As well as the covariant bracket form, a 3+1 decomposition of equation (1.86) may be performed by assuming that F is of the form

$$F = \int n(t)\mathcal{F}(p,q)dt, \tag{1.92}$$

where $n(t)$ is an arbitrary function of time and $\mathcal{F}$ is an arbitrary function of spatial coordinates q and spatial momenta p. Substitution of the functional F (1.92) and action $S[\Gamma]$ (1.88) into the bracket (1.91) yields the result

$$\{F,S\}(\gamma) = \int n(t)(\dot{\mathcal{F}} - \{\mathcal{F},H\}^{(3)})dt = 0, \tag{1.93}$$

where $\{\mathcal{F},H\}^{(3)}$ is the conventional Poisson bracket. The function $n(t)$ is arbitrary, implying that

$$\dot{\mathcal{F}} = \{\mathcal{F},H\}^{(3)}. \tag{1.94}$$

Among the examples considered by Marsden et al. (1986) were the Maxwell equations and the relativistic flat space Vlasov-Maxwell system. We will treat these examples in what follows. For the Maxwell equation system, one may denote the usual four-vector potential by A. In mathematical terminology, the potential is a one-form and we may construct the electromagnetic field tensor F (two-form) from it in the following manner

$$F = dA,$$

i.e.,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.95}$$

where $\partial_\mu = \partial/\partial x^\mu$ and the index takes the values $\mu = 0, 1, 2, 3$. The standard electromagnetic field Lagrangian with minimal coupling to an external current density J^μ is

$$L[A] = \int \mathcal{L} d^4x = \int (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu) d^4x, \tag{1.96}$$

where the Minkowski metric is used to raise and lower indices. One may derive the covariant momentum variables

$$\Pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu}. \tag{1.97}$$

The functional derivatives are defined by analogy with equation (1.90) except that there is a constraint on $\Pi^{\mu\nu}$ due to its antisymmetry

$$\frac{d}{ds} F(\Pi^{\mu\nu} + s\delta\Pi^{\mu\nu})|_{s=0} = \int \frac{\delta F}{\delta \Pi^{\mu\nu}} \delta\Pi^{\mu\nu} d^4x, \tag{1.98}$$

where $\delta\Pi^{\mu\nu}$ is an antisymmetric perturbation. A covariant Poisson bracket may be defined as

$$\{F, G\}_V(A, \Pi) = \int \Big(\frac{\delta F}{\delta A_\mu} \frac{\delta G}{\delta \Pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu} \frac{\delta F}{\delta \Pi^{\mu\nu}} \Big) V^\nu d^4x, \tag{1.99}$$

where V^ν is an arbitrary vector field on space-time and the functionals F and G are dependent on the field variables A_μ and $\Pi^{\mu\nu}$. An electromagnetic action S may be defined as a

covariant analogue of equation (1.88)

$$S[A,\Pi] = \int [\Pi^{\mu\nu} A_{\mu,\nu} - H(A,\Pi)] d^4x, \tag{1.100}$$

where

$$\begin{aligned} H(A,\Pi) &= \frac{1}{4}\Pi_{\mu\nu}\Pi^{\mu\nu} + A_\mu J^\mu \\ &= \Pi^{\mu\nu} A_{\mu,\nu} - \mathcal{L}. \end{aligned} \tag{1.101}$$

It may be shown that Maxwell's equations can be derived from

$$\{F, S\}_V(A,\Pi) = 0, \tag{1.102}$$

for any choice of F and V. One may derive the conditions

$$\frac{\delta S}{\delta \Pi^{\mu\nu}} = 0 \tag{1.103a}$$

and

$$\frac{\delta S}{\delta A_\mu} = 0, \tag{1.103b}$$

which imply

$$\Pi^{\mu\nu} = -(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{1.104a}$$

and

$$\Pi^{\mu\nu}{}_{,\nu} = -J^\mu. \tag{1.104b}$$

Taking into account the definition (1.95), the mathematical identities correspond to the Maxwell equations.

A 3+1 spacetime decomposition of this formalism may be performed (cf. Marsden and Weinstein (1982)). Coordinates are chosen so that the space-time vector field V is

$$V = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}. \tag{1.105}$$

We may, for a closed system, choose $J = 0$ and rewrite the action S as

$$S = \int \left\{ \int [\Pi^{i0} A_{i,0} - \mathcal{H}] d^3x \right\} dt, \tag{1.106}$$

where the Hamiltonian density has the form

$$\mathcal{H} = \frac{1}{2}(\Pi_{ij}\Pi^{ij} + \Pi_{i0}\Pi^{i0}) \tag{1.107}$$

and Latin indices take the values $i = 1, 2, 3$. We will choose a functional of the form

$$F = \int n(t)\bar{F}[A_i, \Pi^{i0}]dt, \tag{1.108}$$

where A_i and Π^{i0} are the respective 3+1 field and conjugate momenta variables. A computation of the bracket (1.102) yields the result

$$\begin{aligned}
\{F, S\}_{\partial/\partial x^0} = \int \left\{ \int \left\{ \frac{\delta \bar{F}}{\delta A_i} \left(A_{i,0} - \frac{\partial \mathcal{H}}{\partial \Pi^{i0}} \right) \right.\right. \\
\left.\left. + \left(\Pi^{i0}{}_{,0} + \frac{\partial \mathcal{H}}{\partial A_i} \right) \frac{\delta \bar{F}}{\delta \Pi^{i0}} \right\} d^3x \right\} n(t) dt \\
= \int \{ \dot{\bar{F}} - \{\bar{F}, \bar{H}\}^{(3)} \} n(t) dt,
\end{aligned} \tag{1.109}$$

where

$$\bar{H} = \int \mathcal{H} d^3x \tag{1.110}$$

and $\{.,.\}^{(3)}$ is the canonical Poisson bracket for the conjugate field variables A_i and Π^{i0}. Due to the arbitrariness of the function $n(t)$, we may conclude that

$$\dot{\bar{F}} = \{\bar{F}, \bar{H}\}^{(3)}. \tag{1.111}$$

It is assumed that $\bar{H}$ is only a function of the conjugate field variables A_i and $\Pi^i = \Pi^{i0}$. This follows by substituting

$$\Pi^{ij} = -(\partial^i A^j - \partial^j A^i) \tag{1.112}$$

into equation (1.107).

We may now examine the case of the relativistic Vlasov-Maxwell equations. A special relativistic particle moves in an external electromagnetic field $F = dA$ in accord with the Lorentz force law

$$\frac{dx^\mu}{d\tau} = u^\mu; \quad \frac{du^\mu}{d\tau} = \frac{e}{m} F^{\mu\nu} u_\nu, \tag{1.113}$$

where e is the electric charge, m is the particle rest mass and τ is the proper time of the particle. The momenta canonically conjugate to x^μ is

$$p_\mu = m u_\mu + \frac{e}{c} A_\mu. \tag{1.114}$$

One may construct the following particle Hamiltonian

$$H = \frac{m}{2} u^\mu u_\mu = \frac{1}{2m}\Big(p^\mu - \frac{e}{c}A^\mu\Big)\Big(p_\mu - \frac{e}{c}A_\mu\Big), \tag{1.115}$$

with the corresponding Hamilton equations that are equivalent to

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} = \frac{p^\mu}{m}; \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu} = \frac{e}{c} u^\nu \frac{\partial A_\nu}{\partial x^\mu}.$$

The Boltzmann plasma distribution is constant along the particle world trajectories

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x^\mu} u^\mu + \frac{e}{c}\frac{\partial f}{\partial p_\mu} u^\nu \frac{\partial A_\nu}{\partial x^\mu} = 0. \tag{1.116}$$

This collisionless Boltzmann equation may be reformulated in terms of the following bracket

$$\{f, H\}_{xp} = 0, \tag{1.117}$$

where

$$\{f, g\}_{xp} = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial g}{\partial x^\mu}\frac{\partial f}{\partial p_\mu}.$$

The basic media field of the plasma is the Boltzmann distribution f. We define the bracket of two functionals F and G of the distribution to have the form

$$\{F, G\}(f) = \int f \Big\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\Big\}_{xp} d^4x d^4p. \tag{1.118}$$

The derivation of this bracket will be given in section 1.4. For now, we may accept it as given without further proof. One may write the plasma action S as

$$S[f] = \int f(x,p)H(x,p)d^4xd^4p, \tag{1.119}$$

such that $\delta S/\delta f = H$. It is a straightforward exercise to prove that the covariant bracket equation

$$\{F, S\}(f) = 0 \tag{1.120}$$

is equivalent to the Vlasov equation (1.116). The relativistic Vlasov-Maxwell system may be analyzed in terms of the set $(A_\mu, \Pi^{\mu\nu}, f)$. One may construct the following covariant Poisson bracket

$$\{F,G\}_V(A,\Pi,f) = \int \left\{ \frac{\delta F}{\delta A_\mu} \frac{\delta G}{\delta \Pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu} \frac{\delta F}{\delta \Pi^{\mu\nu}} \right\} V^\nu d^4x$$
$$+ \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\}_{xp} d^4xd^4p. \tag{1.121}$$

One may define the action S for the Vlasov-Maxwell system to be

$$S[A,\Pi,f] = \int \left(\Pi^{\mu\nu} A_{\mu,\nu} - \frac{1}{4} \Pi_{\mu\nu} \Pi^{\mu\nu} \right) d^4x$$
$$+ \int f(x,p) \frac{1}{2m} \left(p_\mu - \frac{e}{c} A_\mu \right) \left(p^\mu - \frac{e}{c} A^\mu \right) d^4xd^4p. \tag{1.122}$$

The field equations are formulated in terms of the relativistic Poisson bracket

$$\{F, S\}_V(A, \Pi, f) = 0, \tag{1.123}$$

for all choices of F and V. From equation (1.123) we deduce the following identities

$$\frac{\delta S}{\delta \Pi^{\mu\nu}} = 0, \quad \frac{\delta S}{\delta A_\mu} = 0,$$

and

$$\int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta S}{\delta f} \right\}_{xp} d^4xd^4p = 0. \tag{1.124}$$

The mathematical identities (1.124) are quite useful. They yield the relativistic Vlasov-Maxwell equations

$$\frac{\partial f}{\partial x^\mu}u^\mu + \frac{e}{c}\frac{\partial f}{\partial p_\mu}u^\nu\frac{\partial A_\nu}{\partial x^\mu} = 0,$$

$$\partial_\mu F^{\mu\nu} = \frac{e}{c}\int u^\nu f(x,p)d^4p,$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.125}$$

1.4 Derivation of the Plasma Bracket

The plasma bracket was first derived in its present form by Morrison (1980a). However, Gibbons (1980) independently derived it. Also, the work of Iwinski and Turski (1976) should be mentioned in this regard. In the previous section, we dealt primarily with Lie brackets that were constructed from field variables ϕ_A and their conjugate momenta $\Pi_A{}^\mu$. In this section, we wish to consider the derivation of the plasma bracket (1.118). Since f is noncanonical (without a corresponding conjugate momentum), this bracket will differ from the previous field-momentum brackets.

Following the treatment in Morrison (1980a), we will again deal with the Vlasov-Maxwell Hamiltonian system. Morrison (1980a) chose to formulate a Poisson bracket in terms of the noncanonical variables $f_\alpha(\vec{x},\vec{v},t)$, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$, where the Boltzmann distribution of species α is denoted as $f_\alpha(\vec{x},\vec{v},t)$, the electric field intensity is denoted as $\vec{E}(\vec{x},t)$, and the magnetic field intensity is denoted as $\vec{B}(\vec{x},t)$. In rationalized Gaussian units, with the speed of light set to unity, the Vlasov-Maxwell system can be written as

$$\partial_t\vec{B}(\vec{x},t) = -\vec{\nabla}\times\vec{E}(\vec{x},t), \tag{1.126a}$$

$$\partial_t\vec{E}(\vec{x},t) = \vec{\nabla}\times\vec{B}(\vec{x},t) - \sum_\alpha e_\alpha\int_{R_1}\vec{v}f_\alpha(z,t)P(z|\vec{x})dz \tag{1.126b}$$

and

$$\partial_t f_\alpha(z,t) = -\vec{v}\cdot\frac{\partial f_\alpha(z,t)}{\partial\vec{r}} - \frac{e_\alpha}{m_\alpha}\int_{R_2}[\vec{E}(\vec{x},t)+\vec{v}\times\vec{B}(\vec{x},t)]\cdot\frac{\partial f_\alpha(z,t)}{\partial\vec{v}}P(z|\vec{x})d\vec{x}, \quad (1.126c)$$

where the Boltzmann distribution f_α is a function of the phase space variable $z = (\vec{r}, \vec{v})$. In the coupling terms of equations (1.126b) and (1.126c), the operator $P(z|\vec{x}) = \delta(\vec{x} - \vec{r})$ has been used. The regions of integration are defined to be $R_1 \equiv A\times R^3$ and $R_2 \equiv A$, where $R = (-\infty, +\infty)$ and $A \subset R^3$.

One wishes to formulate this system (1.126a)-(1.126c) as

$$\partial\Phi^i/\partial t = [\Phi^i, H], \quad i = 0, 1, ..., 6 \quad (1.127)$$

where Φ^i takes the values

$$\Phi^i = f_\alpha, \quad i = 0 \quad (1.128a)$$

$$= \vec{E}, \quad i = 1, 2, 3, \quad (1.128b)$$

$$= \vec{B}, \quad i = 4, 5, 6, \quad (1.128c)$$

and the Hamiltonian functional is written as

$$H\{\Phi^i\} = \sum_\alpha \int \frac{1}{2}m_\alpha v^2 f_\alpha dz + \int \frac{1}{2}(E^2 + B^2)d\vec{x}. \quad (1.129)$$

The operator $[.,.]$ corresponds to the Poisson bracket. One may assume that the solutions of the Vlasov-Maxwell system exist in a vector space $\omega = \omega_1 \times \omega_2$ that is defined over R. The subspace ω_1 has elements that are functions of z, while the subspace ω_2 has elements that are functions of $\vec{x}$. The operator $P(z|\vec{x})$, appearing in equations (1.126b) and (1.126c), maps elements of one subspace into another. The vector space ω has an associated inner product

$$\langle g|h\rangle = \int_{R_1} g_1 h_1 dz + \int_{R_2} g_2 h_2 d\vec{x}, \quad (1.130)$$

where $g_1 \in \omega_1$ and $g_2 \in \omega_2$ are the two components of $g \equiv (g_1, g_2) \in \omega = \omega_1 \times \omega_2$. Use of an antisymmetric operator O^{ij} on ω and the inner product (1.130) allows one to define a Poisson bracket in the following manner

$$[F, G] = \sum_{\alpha,i,j} \langle \delta F/\delta \chi^i | O^{ij} \delta G/\delta \chi^j \rangle. \tag{1.131}$$

The quantities F and G are elements of Ω, a vector space (over R) of Fréchet differentiable functionals of the functions f_α, $\vec{E}$ and $\vec{B}$. Differentiability is defined with respect to the L^2 norm (1.130). The elements in Ω have the form

$$F\{\chi^i\} = F_1\{\chi^0\} + F_2\{\chi^i\}, \quad i \neq 0, \tag{1.132}$$

where

$$F_\beta\{\chi^i\} = \int_{R_\beta} F_\beta(x_\beta, {\chi_k}^{i_\beta}) dx_\beta, \quad \beta = 1, 2,$$

and

$$x_1 = z, \quad x_2 = \vec{x}, \quad i_1 = 0, \quad i_2 = 1, 2, 3.$$

The subscript k on ${\chi_k}^{i_\beta}$ denotes partial derivatives of a general order k. The Hamiltonian functional (1.129) possesses the form (1.132) with $k = 0$. Arbitrary C^∞ functions of ${\chi_k}^i$ (for all i) are also elements of Ω. In equation (1.131) the quantity $\delta F/\delta \chi^i$ is the functional derivative of F with respect to χ^i.

One may define the functional derivative as

$$(d/d\epsilon) F\{\chi^0(z) + \epsilon w(z)\}|_{\epsilon=0} = \langle \delta F/\delta \chi^0 | w \rangle, \tag{1.133}$$

where, without loss of generality, one may set $i = 0$. It is important to note that both $w(z)$ and $\delta F/\delta \chi^0 \in \omega_1$ and for $i \neq 0$ $\delta F/\delta \chi^i \in \omega_2$. For the purposes of computation, the relations $\delta \chi^0(z)/\delta \chi^0(z') = \delta(z - z')$ and $\delta \chi^i(\vec{x})/\delta \chi^i(\vec{x}') = \delta(\vec{x} - \vec{x}')$ are useful. Functional differentiation of the $k = 0$ Hamiltonian (1.129) will involve only the following three

quantities

$$\delta H/\delta f_\alpha = \frac{1}{2} m_\alpha v^2, \tag{1.134a}$$

$$\delta H/\delta \vec{E} = \vec{E}, \tag{1.134b}$$

and

$$\delta H/\delta \vec{B} = \vec{B}. \tag{1.134c}$$

The bracket (1.131) is (i) antisymmetric $[F, G] = -[G, F]$ and (ii) satisfies the Jacobi identity $[E, [F, G]] + [F, [G, E]] + [G, [E, F]] = 0$, where E, F and $G \in \Omega$. These are the standard properties associated with a Poisson bracket. A Lie algebra is defined as a Poisson bracket together with a vector space (Arnold (1978)). One may define a Hamiltonian system to be a set of partial differential equations with an integral invariant and a bracket satisfying properties (i) and (ii), such that the system may possess the form of equation (1.127), i.e., $\partial_t F = [F, H]$. The antisymmetric operator O^{ij} will be written in the following cosymplectic form

$-\{f_\alpha, \cdot\}$	$-\vec{P}_E^{2\to 1}$	$-\vec{P}_B^{2\to 1}$
$\vec{P}_E^{1\to 2}$	O_3	D
$\vec{P}_B^{1\to 2}$	-D	O_3

where O_3 is a 3×3 array of zeros and $D \equiv [\epsilon_{ijk}(\partial/\partial x_j)]$ (ϵ_{ijk} is the Levi-Civita tensor).

The braces appearing in the upper left corner denote the usual Poisson bracket

$$\{g, h\} = \frac{\partial g}{\partial \vec{r}}\frac{\partial h}{\partial \vec{v}} - \frac{\partial g}{\partial \vec{v}}\frac{\partial h}{\partial \vec{r}} \tag{1.135}$$

which will map elements of ω_1 into ω_1. The lower right-hand block (consisting of D's and O_3's) will map elements of ω_2 into ω_2. The off-diagonal elements

$$\vec{P}_E^{2\to 1} \equiv \frac{e_\alpha}{m_\alpha}\int_{R_2} (\cdot)\frac{\partial f_\alpha}{\partial \vec{v}} P d\vec{x} \tag{1.136a}$$

and

$$\vec{P}_B^{2\to1} \equiv \frac{e_\alpha}{m_\alpha}\int_{R_2}(\cdot)\Big(\frac{\partial f_\alpha}{\partial\vec{v}}\times\vec{v}\Big)Pd\vec{x} \tag{1.136b}$$

will, as seen by their notation, map elements of ω_2 into ω_1. The other off-diagonal elements

$$\vec{P}_E^{1\to2} \equiv \frac{e_\alpha}{m_\alpha}\int_{R_1}(\cdot)\frac{\partial f_\alpha}{\partial\vec{v}}Pdz \tag{1.137a}$$

and

$$\vec{P}_B^{1\to2} \equiv \frac{e_\alpha}{m_\alpha}\int_{R_1}(\cdot)\Big(\frac{\partial f_\alpha}{\partial\vec{v}}\times\vec{v}\Big)Pdz \tag{1.137b}$$

will map elements of ω_1 into ω_2. One may rewrite equation (1.131) in the more explicit form

$$[F,G] = [F,G]_1 + [F,G]_2, \tag{1.138}$$

where

$$\begin{aligned}[F,G]_1 \equiv \sum_\alpha\Big\{&\int_{R_1} f_\alpha\Big\{\frac{\delta F}{\delta f_\alpha},\frac{\delta G}{\delta f_\alpha}\Big\}dz\\ &-\int_{R_1}\frac{\delta F}{\delta f_\alpha}\vec{P}_E^{2\to1}\cdot\Big(\frac{\delta G}{\delta\vec{E}}\Big)dz\\ &-\int_{R_1}\frac{\delta F}{\delta f_\alpha}\vec{P}_B^{2\to1}\cdot\Big(\frac{\delta G}{\delta\vec{B}}\Big)dz\Big\}\end{aligned} \tag{1.139a}$$

and

$$\begin{aligned}[F,G]_2 \equiv \sum_\alpha\Big\{&\int_{R_2}\frac{\delta F}{\delta\vec{E}}\cdot\vec{P}_E^{1\to2}\Big(\frac{\delta G}{\delta f_\alpha}\Big)d\vec{x}\\ &+\int_{R_2}\frac{\delta F}{\delta\vec{B}}\cdot\vec{P}_B^{1\to2}\Big(\frac{\delta G}{\partial f_\alpha}\Big)d\vec{x}\Big\}\\ +\int_{R_2}&\Big(\frac{\delta F}{\delta\vec{E}}\cdot\vec{\nabla}\times\frac{\delta G}{\delta\vec{B}}-\frac{\delta G}{\delta\vec{B}}\cdot\vec{\nabla}\times\frac{\delta F}{\delta\vec{E}}\Big)d\vec{x}.\end{aligned} \tag{1.139b}$$

Using this bracket, one may derive the Vlasov-Maxwell system (1.126a)-(1.126c) from equation (1.127). The first term of equation (1.139a) will yield the Vlasov equation without the coupling term involving electromagnetic field variables. The final term in equation (1.139b) yields the vacuum Maxwell equations. The other terms in equation (1.139a) and (1.139b) correspond to field-distribution couplings.

One may study the Vlasov-Poisson Hamiltonian system by neglecting terms in equation (1.138) and (1.129) that involve $\vec{B}$. The resultant expression

$$[F,G] = \sum_\alpha \int_{R_1} f_\alpha(z)\Big\{\frac{\delta F}{\delta f_\alpha}, \frac{\delta G}{\delta f_\alpha}\Big\} dz \tag{1.140}$$

is obtained by writing the electric field $\vec{E}$ as a functional of the distribution f_α. The corresponding Vlasov equation will be

$$\partial_t f_\alpha = -\{f_\alpha, H_p{}^\alpha\}, \tag{1.141}$$

where

$$H_E\{f_\alpha\} = \sum_\alpha \Big(\int_{R_1} H_1{}^\alpha(z) f_\alpha(z) dz \tag{1.142a}$$

$$+\frac{1}{2}\int_{R_1}\int_{R_1} f_\alpha(z) f_\alpha(z') H_2{}^\alpha(z|z') dz dz'\Big),$$

$$H_1{}^\alpha = \frac{1}{2} m_\alpha v^2 \tag{1.142b}$$

and

$$H_2{}^\alpha = e_\alpha{}^2/|\vec{r} - \vec{r}'|. \tag{1.142c}$$

The particle Hamiltonian takes the form $H_p{}^\alpha = \delta H_E/\delta f_\alpha$.

CHAPTER 2
NEWTONIAN COSMOLOGY AND THE VLASOV-MAXWELL SYSTEM

A Newtonian cosmology (Peebles (1980)) is typically viewed as the infinite radius limit of a homogeneous and isotropic sphere of matter that expands homologously with the expansion rate governed by a time-dependent scale factor $a(t)$. We may formulate a collisionless Boltzmann, that is, gravitational Vlasov equation to describe the evolution of a one-particle distribution function f for such a scenario. We may consider so-called "equilibrium" solutions f_o which correspond to expanding, steady state distributions (Peebles (1980); Bisnovatyi-Kogan and Zel'dovich (1970)). This is as straightforward as the construction of equilibria for isolated self-gravitating systems. However, for an isolated system, the equilibrium distributions f_o lack explicit time-dependence. This facilitates the use of energy arguments to study the stability problem (Antonov (1961); Lynden-Bell and Sanitt (1969); Kandrup and Sygnet (1985)).

Formulated in an inertial frame, cosmological so-called "equilibria" possess an explicit time-dependence which renders the standard energy arguments inadmissable. There is a way, however, to reformulate the problem so that cosmological "equilibria" possess no explicit time-dependence. This is done by reformulating the Vlasov description in terms of the noninertial average comoving frame. In this frame, the cosmological "equilibria" possess no explicit time-dependence. Under such a transformation, the Hamiltonian will acquire an explicit time-dependence. In spite of this, a cosymplectic structure may still be found and an energy criterion for linear instability can be given. It should be remembered that it is not necessary to use energy arguments to study the problem of stability for cosmolog-

ical equilibria. Instead, we may resort to a direct analysis of the linearized perturbation equations to determine the time-dependence of quantities such as the perturbed density $\delta\rho$. However, because energy arguments are extremely useful in the study of the stability of isolated equilibria, they may be amenable to the study of Newtonian cosmological equilibria. The technique used here to extract a nontrivial stability criterion is quite general in the sense that, in the average co-moving frame, the Hamiltonian acquires an explicit time-dependence and the equilibrium corresponds to an energy extremum. Therefore, the general techniques may be applicable to other physical problems.

2.1 The Transformed Vlasov-Poisson Equation

The distribution function $f(R^a, P_a, m, T)$ associated with the inertial frame may be defined as the number density of particles of mass m with momentum P_a at the point R_a for the time T. The gravitational Vlasov equation

$$\frac{\partial f}{\partial T} + \frac{P^a}{m}\frac{\partial f}{\partial R^a} - m\frac{\partial \Phi}{\partial R^a}\frac{\partial f}{\partial P_a} = 0 \tag{2.1}$$

determines the evolution of the distribution f. The gravitational potential Φ is self-consistently determined by the expression

$$\Phi(\vec{R}, T) = -G\int d^3R' d^3P' dm' \frac{m' f(\vec{R}', \vec{P}', m', T)}{|\vec{R} - \vec{R}'|}. \tag{2.2}$$

We wish to transform to a new set of variables $\{r^a, p_a, m, t\}$ from the original variables $\{R^a, P_a, m, T\}$ via a time-dependent canonical transformation of the form

$$r^a = a(T)^{-1}R^a, \quad p_a = a(t)[P_a - mH(t)\delta_{ab}R^b]$$

and

$$dt = a(t)^{-2}dT \tag{2.3}$$

where

$$H \equiv d(\log a)/dT,$$

and a is an arbitrary function of time. The transformed Vlasov-Poisson equation will be of the form

$$\frac{\partial f}{\partial t} + \frac{p^a}{m}\frac{\partial f}{\partial r^a} - ma(t)\left[\frac{\partial \Psi}{\partial r^a} - \Omega^2 r^a\right]\frac{\partial f}{\partial p_a} = 0, \tag{2.4}$$

where

$$\Psi(\vec{r},t) = -G\int d^3r' d^3p' dm' \frac{m' f(\vec{r}',\vec{p}',m',t)}{|\vec{r}-\vec{r}'|}$$

and

$$\Omega^2 = -a^2(d^2a/dT^2).$$

Equation (2.4) holds true for an arbitrary choice of $a(T)$. However, an appropriate choice of a will cause the steady state equilibria to take a simple form.

In this context, we may choose $a(T)$ to satisfy the equation

$$a^2\frac{d^2a}{dT^2} = -\frac{4\pi G}{3}\rho(T)a^3 = -\frac{4\pi G}{3}\rho_o, \tag{2.5}$$

where $\rho(T)$ is the average mass density of the Universe at time T. In the average co-moving frame, the quantity $\rho_o = a^3\rho$ represents the constant, time-independent density of matter in the Universe. For this choice of $a(T)$ the physical peculiar velocity

$$V^a = \frac{p^a}{m} - HR^a$$

will decay as $1/a$. It follows that the momentum p_a is conserved. We may construct a time-independent solution $f_o(p_a)$ for the transformed Vlasov equation (2.4) from this constant of the motion. With the assumption that f_o is independent of r^a, we may deduce that

$$\frac{\partial \Psi}{\partial r^a}(\vec{r},t) = -G\frac{\partial}{\partial r^a}\int d^3r' d^3p' dm' \frac{m' f_o(\vec{r}',\vec{p}',m',t)}{|\vec{r}-\vec{r}'|}$$

$$= -G\frac{\partial}{\partial r^a}\int d^3r' \frac{\rho_o}{|\vec{r}-\vec{r}'|} = \frac{4\pi G}{3}\rho_o r^a. \tag{2.6}$$

Therefore, we can write $\partial\Psi/\partial r^a = \Omega^2 r^a$. It follows that the $\partial f/\partial p_a$ force term in the Vlasov equation (2.4) will vanish. The $\partial f/\partial r^a$ term will vanish if spatial homogeneity is assumed. Consequently, we find that the distribution corresponds to a time-independent equilibrium $\partial f/\partial t \equiv 0$.

2.2 The Vlasov-Poisson Hamiltonian Structure

Hamiltonian formulations of the gravitational Vlasov-Poisson and Vlasov-Einstein equations, for isolated self-gravitating systems, have already been presented. By direct analogy, the Vlasov equation (2.4) for the Newtonian cosmological system can also be shown to admit a Hamiltonian formulation. It is necessary (Morrison (1980a); Morrison and Pfirsch (1990)) to identify a Lie bracket $[A, B]$ and a Hamiltonian function H in order that the Vlasov-Poisson system may be written as

$$\frac{\partial f}{\partial t} = -[H, f]. \tag{2.7}$$

In accord with the ordinary Vlasov-Poisson system, we can define the bracket operation as

$$[A, B] = \int d\Gamma f \left\{ \frac{\delta A}{\delta f}, \frac{\delta B}{\delta f} \right\}, \tag{2.8}$$

where $\{a, b\}$ is the ordinary Poisson bracket, $\delta/\delta f$ is a functional derivative of the Boltzmann distribution and $d\Gamma = d^3x d^3p dm$ is a differentiable phase space volume element. It is straightforward to verify that the bracket is antisymmetric and satisfies the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{2.9}$$

and therefore constitutes a *bona fide* Lie bracket. Consequently, the evolution generated by any Hamiltonian H and this Lie bracket will be symplectic. We may view the evolution as a

generalized canonical transformation in an infinite-dimensional phase space of distribution functions. We shall select the "natural" Hamiltonian

$$H = \int d\Gamma \frac{p^2}{2m} f - \frac{Ga(t)}{2} \int d^3r \int d^3r' \frac{[\rho(r) - \rho_o][\rho(r') - \rho_o]}{|\vec{r} - \vec{r}'|}, \tag{2.10}$$

to generate the transformed Vlasov equation (2.4). In equation (2.10) we have

$$\rho(\vec{r}, t) = \int d^3p dm m f(\vec{r}, \vec{p}, m, t)$$

and ρ_o satisfies equation (2.5).

We calculate $\delta H / \delta f = E$ to have the form

$$E = \frac{p^2}{2m} + a(t) m \Psi + Ga(t) \int d^3r' \frac{\rho_o}{|\vec{r} - \vec{r}'|}. \tag{2.11}$$

The Vlasov equation (2.4) is then generated from equation (2.7) through this choice of Hamiltonian H

$$\frac{\partial f}{\partial t} = \left\{ \frac{\delta H}{\delta f}, f \right\} = \{E, f\}. \tag{2.12}$$

Other choices, besides equation (2.10), for the Hamiltonian are permitted. We may, for example, choose

$$H' = \int d\Gamma \frac{p^2}{2m} f - \frac{Ga(t)}{2} \int d\Gamma \int d\Gamma' \frac{m f(r, p, m, t) m' f(r', p', m', t)}{|\vec{r} - \vec{r}'|}$$
$$- a(t) \int d\Gamma \frac{m \Omega^2 r^2}{2} f, \tag{2.13}$$

where $\Omega^2 = 4\pi G \rho_o / 3$. This Hamiltonian differs from equation (2.10) by a function independent of f and only dependent on time. The choice of this Hamiltonian is disadvantageous, however, because it remains explicitly time-independent even for a homogeneous equilibrium. Also, for such an equilibrium, the energy

$$E' \equiv \frac{\delta H'}{\delta f} = \frac{p^2}{2m} + a(t) m \Psi - \frac{a(t) m \Omega^2 r^2}{2} \tag{2.14}$$

is different from the "natural" energy $E = p^2/2m$ in that it possesses an overall function of time. It is generally known that, in accord with conservation of phase, the Vlasov equation possesses an infinite number of constraints. If the distribution function behaves properly "at infinity" then we may integrate by parts to prove that the numerical value of an arbitrary constraint

$$C = \int d\Gamma \chi(f) \tag{2.15}$$

is conserved. It follows that the time derivative of C vanishes

$$\frac{dC}{dt} = [C, H] \equiv 0 \tag{2.16}$$

(To validate the integrations by parts it is sufficient to assume that f_o is differentiable and, as $r \to \infty$, independent of r with an isotropic velocity distribution. Although they are not physically intuitive, periodic boundary conditions could also be employed as well as other, less restrictive requirements as $r \to \infty$). It is not possible to define a unique Hamiltonian because of the existence of these Casimir constraints. The Hamiltonian H may be renormalized by adding a constraint C which will, however, leave the Vlasov equation (2.12) unaffected because

$$[C, f] = \left\{ f, \frac{\partial \chi}{\partial f} \right\} = 0. \tag{2.17}$$

The Casimir constraints severely restrict the evolution of the Boltzmann distribution f as dictated by the Vlasov equation. These Casimirs (Morrison (1980a); Morrison and Pfirsch (1990); Kandrup (1990, 1991a); Kandrup and Morrison (1993)) constrain the evolution to an infinite-dimensional hypersurface of the infinite-dimensional phase space. The hypersurface is defined by the constancy of all the constraints. It follows that, when dealing with the problem of linear stability, we should only consider perturbations of the distribution δf that preserve the constancy of the constraints. Only perturbations that propagate dynamically

are relevant to the study of linear stability. We may, in a straightforward fashion, find the generic perturbation of the distribution that will conserve all of the constraints. It can be shown (Kandrup (1990); Morrison (1980a); Bartholomew (1971); Morrison and Pfirsch (1990)) that δC will vanish if the perturbation δf is generated from the equilibrium f_o through a canonical transformation. Therefore, the perturbation must be of the form

$$f_o + \delta f = \exp(\{g, .\})f_o = f_o + \{g, f_o\} + (1/2!)\{g, \{g, f_o\}\} + ..., \tag{2.18}$$

with a generating function g. It is simple, using the dynamically accessible perturbation δf of equation (2.18), to prove that the first variation of the Hamiltonian δH vanishes. This implies that all equilibria are extrema of the Hamiltonian (2.10). We can prove this by first calculating the expression

$$\begin{aligned}\delta^{(1)}H &= \int d\Gamma \delta^{(1)} f \left[\frac{p^2}{2m} - Gma(t)\Big(\int d\Gamma' \frac{m' f_o(x', p', m', t)}{|\vec{r} - \vec{r}'|} - \int d^3x' \frac{\rho_o}{|\vec{r} - \vec{r}'|}\Big)\right] \\ &= \int d\Gamma \delta^{(1)} f E \end{aligned} \tag{2.19}$$

and then substituting in the phase-preserving perturbation $\delta^{(1)} f = \{g, f_o\}$. The net result is

$$\delta^{(1)}H = \int d\Gamma \{g, f_o\} E = -\int d\Gamma \{E, f_o\} g = 0,$$

where, for an equilibrium,

$$\frac{\partial f_o}{\partial t} = \{E, f_o\} = 0.$$

We now consider the second variation of the Hamiltonian $\delta^{(2)}H$. We may calculate the following expression

$$\begin{aligned}\delta^{(2)}H &= \frac{1}{2}\int d\Gamma \delta^{(2)} f E \\ &-\frac{Ga(t)}{2}\int d\Gamma \int d\Gamma' \frac{m\delta^{(1)} f(x, p, m, t) m' \delta^{(1)} f(x', p', m', t)}{|\vec{r} - \vec{r}'|}.\end{aligned} \tag{2.20}$$

Substituting $\delta^{(1)}f$ and $\delta^{(2)}f$, as given by equation (2.18), into this expression yields, after an integration by parts, the final expression

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma\{g,f_o\}\{g,E\} - \frac{Ga(t)}{2}\int d\Gamma \int d\Gamma' \frac{m\{g,f_o\}m'\{g',f_o'\}}{|\vec{r}-\vec{r}'|}. \tag{2.21}$$

This expression can be rewritten in a simpler form. Since the equilibrium f_o is a function of p_a and m, we may write the Poisson bracket as

$$\{g,f_o\} = \frac{\partial g}{\partial x^a}\frac{\partial f_o}{\partial p_a}.$$

The overall homogeneity of the distribution makes it useful to utilize the Fourier transformed generating function

$$g(k^a,p_a,t) = \frac{1}{(2\pi)^{3/2}}\int d^3x e^{-i\vec{k}\cdot\vec{x}} g(x^a,p_a,t). \tag{2.22}$$

Using the Poisson bracket associated with overall homogeneity and the transform (2.22) will allow one to rewrite the second variation of the Hamiltonian as

$$\begin{aligned}\delta^{(2)}H = &-\frac{1}{2}\int d^3k \int d^3p |g(k,p,m,t)|^2(\vec{k}\cdot\vec{u})\left(\vec{k}\cdot\frac{\partial f}{\partial \vec{p}}\right)\\ &-2\pi Ga(t)\int d^3k\int d^3p\int d^3p'\left[mg(k,p,m,t)\frac{\vec{k}}{|\vec{k}|}\cdot\frac{\partial f}{\partial\vec{p}}\right]\\ &\times\left[m'g(k,p',m',t)\frac{\vec{k}}{|\vec{k}|}\cdot\frac{\partial f'}{\partial\vec{p}'}\right],\end{aligned} \tag{2.23}$$

with

$$\vec{u} = \vec{p}/m.$$

If we take f_o to be an isotropic velocity distribution then this expression can be rewritten in a simpler format. For an isotropic velocity distribution, the equilibrium f_o depends on the momentum p_a through the square of the magnitude p^2. Consequently, one can write

$$\{g,f_o\} = F_E\vec{u}\cdot\frac{\partial g}{\partial\vec{r}},$$

where, for an equilibrium, $E = p^2/2m$ and $F_E = \partial f_o/\partial E$. Therefore, we write the second order perturbation of H as

$$\delta^{(2)}H = \frac{1}{2}\int d^3k \int d^3p(-F_E)|g(k,p,m,t)\vec{k}\cdot\vec{u}|^2$$

$$-2\pi Ga(t)\int d^3k \int d^3p \int d^3p' \Big[mg(k,p,m,t)\frac{\vec{k}\cdot\vec{u}}{|\vec{k}|}F_E\Big]$$

$$\times\Big[m'g(k,p',m',t)\frac{\vec{k}\cdot\vec{u'}}{|\vec{k}|}F'_E\Big]. \tag{2.24}$$

2.3 Stability of Collisionless Vlasov-Poisson Equilibria

A simple stability criterion can be obtained from equation (2.23) or equation (2.24). The time derivative

$$\frac{d\delta^{(2)}H}{dt} = -\frac{G}{2}\frac{da}{dt}\int d\Gamma \int d\Gamma' \frac{m\{g, f_o\}m'\{g', f_o'\}}{|\vec{r}-\vec{r'}|} \tag{2.25}$$

is negative semi-definite because, for an expanding universe, $da/dt > 0$. Therefore, in the average co-moving frame, any initial phase-preserving perturbation undergoes an intrinsically dissipative evolution. The perturbed energy $\delta^{(2)}H$ will decrease after the initial perturbation. One may consider the case where $\delta^{(2)}H(t_o) < 0$, where t_o is the time of the initial perturbation. Since the evolution is dissipative the quantity $\delta^{(2)}H(t)$ can only become more negative. The system will continue to evolve away from equilibrium with the magnitude of the second order perturbation $|\delta^{(2)}H|$ increasing. Consequently, the system is linearly unstable. A perturbation of a certain minimum wavelength will cause the second negative contribution of equations (2.23) and (2.24) to have a magnitude greater than the first term which is of indefinite sign. This minimum wavelength will be of the order of magnitude of the Jeans length. An order of magnitude estimate indicates that the first and second terms of equation (2.24) will be comparable in magnitude for a characteristic perturbation wave number $k^2 \sim Ga(t)\rho_o/\bar{v}^2$, where $\bar{v}$ is a characteristic velocity and ρ_o is

the co-moving density. This estimate is performed by assuming that both the unperturbed distribution f_o and the generating function g are well behaved, and that the derivative F_E is everywhere negative. One can calculate the physical wavelength $\lambda \sim (k/a)^{-1}\,(\bar{V}^2/G\rho)^{1/2}$, which corresponds to the Jeans length, by utilizing the expressions for the physical density $\rho = \rho_o/a^3$ and the physical peculiar velocity $V = a^{-1}v$. The energy must be negative for wavelengths greater than the Jeans length. Specifying the form of f_o enables us to obtain a better estimate of the physical wavelength. Therefore, the derivation of the Jeans instability via an energy argument given here is rigorous.

By making certain assumptions, it can be shown that f_o is unstable to perturbations with wavelength much shorter than the Jeans length. One example of this occurs for an equilibrium f_o with an isotropic distribution of velocities and a derivative F_E that is not everywhere negative. As a specific case of this, we consider the distribution $f_o = E^b \exp(-\beta E)$, where b is a positive constant. Substituting into equation (2.24) a generating function g with nontrivial k dependence that is sharply peaked about the velocities where F_E is positive leads to an instability. In the previous example, this occurs as $v^2 \to 0$. Therefore, for this example, both terms in equation (2.24) will be negative leading to a second order energy perturbation $\delta^{(2)}H$ and a corresponding time derivative $d\delta^{(2)}H/dt$ which are both negative. Another example involves any plane-symmetric equilibrium $f_o(\vec{p_a}) = f_o(\vec{p_a}^2)$. Such an equilibria is unstable if there exists a range of velocities and a direction for which the derivative $\partial f/\partial {p_a}^2$ is positive. If we choose a k-vector, $\vec{k} = k_a\hat{a}$, aligned in an appropriate direction such that

$$-|g(k,p,m,t)|^2(\vec{k}\cdot\vec{u})\left(\vec{k}\cdot\frac{\partial f}{\partial \vec{p}}\right) = -|g(k,p,m,t)|^2|k_a|^2 u_a \frac{\partial f}{\partial p_a}$$

$$= -2m|g(k,p,m,t)|^2|k_a|^2|u_a|^2 \frac{\partial f}{\partial {p_a}^2}, \tag{2.26}$$

then, for $\partial f/\partial {p_a}^2 > 0$, this expression will be intrinsically negative. However, if there does not exist a direction in velocity space for which $\partial f/\partial {p_a}^2$ is positive, then the integrand of the first term integral in equation (2.23) is positive and the equilibria will be stable when subjected to perturbations of sufficiently short wavelengths. For example, if we choose a k-vector $\vec{k} = k_a \hat{a}$

$$-|g|^2(\vec{k}\cdot\vec{u})\left(\vec{k}\cdot\frac{\partial f}{\partial \vec{p}}\right) = -|g|^2|k_a|^2 u_a \frac{\partial f}{\partial p_a} = -2m|g|^2|k_a|^2|u_a|^2 \frac{\partial f}{\partial {p_a}^2} > 0. \qquad (2.27)$$

It follows that the aforementioned equilibria will be stable for all perturbations of characteristic wavelength much shorter than the Jeans length provided that they are monotonically decreasing functions of speed for all directions of velocity space. Consequently, we do not allow for population inversions.

This argument does not imply that a negative energy perturbation of characteristic length scale R will demonstrate instability associated with a natural time of $t_D = R/\bar{V}$. Instead, the magnitude of $\delta^{(2)}H$ will increase with a time scale determined by the variation of $a(t)$. Consequently, the Hubble time scale t_H sets a lower bound on the growth time of the instability. Even if we consider the case of a time-independent energy $\delta^{(2)}H$, e.g., perturbations of static equilibria possessing compact support or, as an approximate case, a cosmological setting with time scale $\ll t_H$, a dynamical instability with a time scale $t_D(R)$ could be implied for a negative energy perturbation $\delta^{(2)}H < 0$. However, it is not certain that an instability will follow from a time-independent $\delta^{(2)}H < 0$ because none of the negative energy perturbations must necessarily be coupled to the positive energy perturbations. A sufficient, but not necessary (Kandrup (1991a)), condition of stability, for a time-independent energy with a static setting, occurs when $\delta^{(2)}H > 0$.

We now consider a straightforward counterexample. The Hamiltonian of a two degree of freedom system

$$H = \frac{1}{2}({p_1}^2 + {\omega_1}^2{q_1}^2) - \frac{1}{2}({p_2}^2 + {\omega_2}^2{q_2}^2) \tag{2.28}$$

possesses an equilibrium solution $q_1 = p_1 = q_2 = p_2 = 0$ which is stable if the frequencies are constant, yet becomes unstable if $d{\omega_2}^2/dt > 0$. The instability resulting from dissipation, through the effects of gravitational radiation, has already been studied for rotating perfect fluid stars (Friedman and Schutz (1978)) and generic rotating, axisymmetric equilibrium solutions of the gravitational Vlasov-Poisson system for isolated systems. The energy argument dealing with instability that is given here is directly analogous to the previously studied cases. However, the Newtonian cosmology is intrinsically negative, while, in the previous cases, an additional source of dissipation, such as gravitational radiation, must be invoked to bring about energetic instability. From a theoretical viewpoint, this is due to the fact that the equilibria are static in the conformal-but not true-sense.

2.4 Introduction to the Curved Space Vlasov-Maxwell System

At this point we shall consider the Hamiltonian formulation of the Vlasov-Maxwell system in a curved background space-time. This exercise will serve as a springboard to a similar treatment of the Vlasov-Einstein system which will be dealt with in chapter 3. The Vlasov-Maxwell system couples the Maxwell equations, governing the behavior of an electromagnetic field, together with the Boltzmann distribution f which serves as a source for the electromagnetic field. The Boltzmann distribution f is coupled to the electromagnetic field in the sense that it is influenced by it while serving as its source. The evolution of the distribution f is self-consistently determined by the collisionless Boltzmann (Liouville) equation. If we take the Boltzmann distribution f to be a function of the space-time coordinates x^α and the physical (*not* canonical) momentum P^α, then it may be defined with respect to the

tangent bundle corresponding to the background space-time. Therefore, the distribution f satisfies the following Liouville equation (cf. Israel (1972) and Stewart (1971))

$$\frac{P^\alpha}{m}\frac{\partial f}{\partial x^\alpha} + (eF^\alpha{}_\mu P^\mu - \Gamma^\alpha{}_{\mu\nu}P^\mu P^\nu)\frac{\partial f}{\partial P^\alpha} = 0, \tag{2.29}$$

where $\Gamma^\alpha{}_{\mu\nu}$ is the Christoffel symbol defined by the space-time metric $g_{\mu\nu}$, and the Faraday tensor $F_{\mu\nu}$ satisfies the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = J^\nu \equiv e\int(-g)^{-1/2}d^4P\frac{P^\nu}{m}f$$

and

$$\nabla_\mu{}^*F^{\mu\nu} = 0, \tag{2.30}$$

where $*$ denotes a dual tensor. We recover the electrostatic Vlasov-Poisson system by taking the Newtonian limit $c \to \infty$. For any dynamical system, there are a variety of methods for implementing a Hamiltonian formulation. We shall utilize the most general technique, which is to proceed at a formal algebraic level. Accordingly, we shall work with a phase space γ and a Hamiltonian H defined with respect to it. Also, a Lie bracket $\langle F, G\rangle$ which acts on functionals F and G of the phase space, will be used. It will be shown that, with suitable choices for the Hamiltonian H and Lie bracket, the Vlasov-Maxwell system may be formulated in terms of the constraint equations $\partial_t F = \langle F, H\rangle$, where F is an arbitrary functional. From classical mechanics, the standard Poisson bracket equation $\partial_t F = \{F, H\}$, and its associated Hamiltonian H, demonstrate that the dynamics in the six-dimensional phase space (x^i, p_i) is generated through a canonical transformation. Likewise, the Lie bracket equation $\partial_t F = \langle F, H\rangle$ implies that the dynamics of F in the phase space γ is generated by a type of generalized canonical transformation. However, this cannot be a canonical transformation because γ does not have canonical coordinates.

Initially, we might presume to take the phase space γ as being constructed from spatial coordinates x^i and momenta p_i, and electromagnetic coordinates, i.e., the vector potential A_i and conjugate momenta Π_i. Unfortunately, this presumption is incorrect. It would be appropriate to select a six-dimensional phase space (x^i, p_i) for studying the dynamics of a Boltzmann distribution in a fixed electromagnetic field. If we take ε as representing the canonical particle energy, then, in this case, the Liouville equation can be written as $\partial_t f = \{\varepsilon, f\}$. Conversely, the phase space (A_i, Π^i) would be employed for the examination of the dynamics of an electromagnetic field with respect to a fixed current source J^i. Taking the electromagnetic Hamiltonian (2.41), with an additional minimal coupling $\int (-g)^{-1/2} d^3x J_i A^i$, enables one to generate canonically the Maxwell equations corresponding to the problem. For the Vlasov-Maxwell system, however, one is not free to hold either the electromagnetic field or the Boltzmann distribution fixed. Our Hamiltonian description must take into account the reciprocity of the interaction between field and source, i.e., the electromagnetic field controls the dynamics of the Boltzmann distribution f which acts as the source of the electromagnetic field.

In this case, we will take f to be a dynamical variable in the γ phase space (A_i, Π^i, f). However, the γ phase space is *not* canonical in contrast to the six-dimensional (x^i, p_i) phase space or the infinite dimensional (A_i, Π^i) phase space. One can separate the electromagnetic variables into canonical pairs $A_i(x^j)$ and $\Pi^i(x^j)$, but such a division of the distribution f into canonical pairs is not straightforward. If we attempt such a decomposition, then we must explicitly implement the infinite number of constraints associated with the Vlasov-Maxwell system because of conservation of phase. These constraints restrict the evolution to a still infinite-dimensional hypersurface of the infinite-dimensional γ space. The distribution f may be decomposed into canonical pairs only in this hypersurface of the phase

space. Since the cosymplectic structure is degenerate from the perspective of the full γ phase space, this should be obvious. It remains valid, however, to use a noncanonical phase space Hamiltonian formulation, but, at some point in the analysis, the constraints corresponding to phase conservation must be explicitly implemented. Since the dynamics of the Vlasov-Maxwell system are generated by a generalized canonical transformation, it is possible to intuitively understand the evolution of the system. As for the Newtonian cosmology, a Hamiltonian formulation allows the problem of stability to be studied through the use of energy arguments. We wish to examine the energetic behavior of phase-preserving perturbations $\delta X = (\delta A_i, \delta\Pi^i, \delta f)$ of the equilibria $X_o = ({A_i}^o, {\Pi_o}^i, f_o)$ associated with the Vlasov-Maxwell system. The interaction between electromagnetic field and Boltzmann distribution occurs on a fixed background space-time which possesses a timelike Killing field. It is straightforward to deduce a Hamiltonian formulation for the flat space electrostatic Vlasov-Poisson system.

The electrostatic system has no radiative degrees of freedom and, with a choice of suitable boundary conditions, the electric field $E^i(x^j)$ at time t can be written as a functional of the Boltzmann distribution f at the same time instant. We must find a suitable Green function corresponding to the Poisson equation. Therefore, the only dynamical variable of the system is the Boltzmann distribution f. So, the γ phase space reduces to the phase space of distribution functions f. The Hamiltonian will have functional dependence $H = H[f]$ and the Lie bracket $\langle F, G \rangle$ will act on functionals $F[f]$ and $G[f]$ of the phase space.

This Hamiltonian structure for the electrostatic Vlasov-Poisson system has been examined by Morrison (1980a, b) and Gibbons (1981). It is simple, for the Vlasov-Poisson system, to find phase-preserving perturbations (Gardner (1963); Bartholomew (1971); Morrison (1987); Morrison and Pfirsch (1989, 1990, 1992)) δf that yield energy extrema $\delta^{(1)} H = 0$.

We can deduce nontrivial energy stability criteria by calculating the second order variation $\delta^{(2)}H$. One example of a Vlasov-Poisson system would be a neutral plasma. Such a plasma consists of a fixed homogeneous, positive background of heavy ions interspersed with a gas of light electrons.

On average, in a neutral plasma, the negative and positive charges cancel out and there is no external field present. A homogeneous and isotropic equilibrium electron distribution can be described by a function $f_o(E)$, where $E = p^2/2m$ is the particle energy. If the derivative $\partial f_o/\partial E = F_E$ is everywhere negative, then $\delta^{(2)}H$ is positive for all phase-preserving perturbations δf which implies that the equilibrium is linearly stable. However, if, for some range of energies, F_E exhibits a population inversion, then there exist perturbations δf such that $\delta^{(2)}H < 0$, which, however, does not necessarily indicate a linear instability (Holm, Marsden, Ratiu, and Weinstein (1985); Bernstein (1958); Gardner (1963); Penrose (1963)).

Similar reasoning can be employed to derive a Hamiltonian formulation for the gravitational Vlasov-Poisson system (Kandrup (1990)). For self-gravitating systems, there are no homogeneous, static equilibria and the sign of $\delta^{(2)}H$ becomes more ambiguous because of the attractive nature of the gravitational interaction. For the gravitational Vlasov-Poisson system there exists (Sygnet, Des Forets, Lachize-Rey, and Pellat (1984); Kandrup and Sygnet (1985); Kandrup (1991a)) a stability theorem which is a direct analog of a similar theorem for isotropic electrostatic equilibria. For the gravitational Vlasov-Poisson system, any spherically symmetric equilibria $f(E, m)$ that is a monotonically decreasing function of the particle energy $E = p^2/2m + m\Phi_o$, for all values of the mass species m, is necessarily stable. The symbol Φ_o represents the gravitational potential which is a functional of the Boltzmann distribution f_o. The flat space Vlasov-Maxwell system has had its corresponding Hamiltonian structure studied through a variety of methods (Iwinski and Turski

(1976); Morrison (1980a); Marsden and Weinstein (1982)). The stability criteria for the flat space Vlasov-Maxwell system are directly analogous to those of the aforementioned Vlasov-Poisson system. Therefore, for a dynamically accessible perturbation, $\delta^{(1)}H = 0$, and the sign of the second variation $\delta^{(2)}H$ provides a useful criterion for stability.

The Hamiltonian structure of an electromagnetic fluid in special relativity, has, to a certain extent, been analyzed (Holm (1987)). We wish to derive the Hamiltonian structure for the general curved space Vlasov-Maxwell system. The Lie bracket utilized by Marsden and Weinstein (1982) for the case of a flat space Vlasov-Maxwell Hamiltonian system will be used here. However, unlike Marsden and Weinstein, the formulation will be given, with the exception of equations (2.59) and (2.60), in terms of the conjugate electromagnetic variables A_i and Π^i and the canonical particle momentum p_i. It is possible (Marsden and Weinstein (1982)) to choose the electric and magnetic field densities E^i and B^i and the contravariant physical momentum P^i as alternate variables. However, this choice, for a curved space-time background, yields an unnecessarily complex formulation. For example, one can make a comparison between equation (2.45) and equations (2.59) and (2.60) to see this.

The total Hamiltonian consists of an electromagnetic piece H_{EM} and a matter contribution H_M. The flat space electromagnetic Hamiltonian, corresponding to the canonical formulation, is generalized to the setting of a curved background space-time. The matter Hamiltonian used by Marsden and Weinstein (1982) is different from the one used here. If we write the canonical formalism in terms of an ADM 3+1 split into space and time, and select momentum variables corresponding to particle mass m and spatial momentum p_i, rather than four-momenta p_α (or P_α or P^α), then, for simplicity of application, we should use the matter Hamiltonian H_M discussed here, but not the form considered by Marsden

and Weinstein (1982). With these choices for the Hamiltonian components it will become obvious that (i) the Vlasov equation is generated by the canonical particle energy ε and (ii) the current J^i is given by the functional derivative $\delta H_M/\delta A_i$.

2.5 The Vlasov-Maxwell Hamiltonian Formulation

In analyzing the Hamiltonian structure of the curved space Vlasov-Maxwell system, it is convenient to work with the canonical coordinates (x^μ, p_μ). The canonical momentum p_μ can be written in terms of the physical momentum P^μ and vector potential A_μ as $p_\mu = g_{\mu\nu}P^\nu + eA_\mu$. For a fixed particle mass m, the mass shell constraint $g_{\mu\nu}P^\mu P^\nu = g^{\mu\nu}P_\mu P_\nu = -m^2$ allows one to fix the value of any four-momentum component in terms of the remaining three. The expression for the fundamental phase space element is

$$((-g)^{1/2}d^4x)\left(\frac{-1}{(-g)^{1/2}}d^4p\right) = -d^4xd^4p.$$

The explicit 3+1 split into space and time, along with the mass shell constraint, makes it convenient to choose p_i and m as momentum variables. This choice allows one to deduce the expression $-mdm = g^{t\mu}P_\mu dP_t = P^t dP_t = (p^t - eA^t)dp_t$. One calculates that $dp_t = -mdm/P^t$, which is used to rewrite the covariant volume element as

$$-d^4xd^4p \rightarrow -\frac{P^t}{m}d\tau d^3xd^3p\left(-\frac{mdm}{P^t}\right) = d\tau d^3xd^3pdm = d\tau d\Gamma dm, \qquad (2.31)$$

where τ represents the invariant proper time and $d\Gamma$ represents the invariant 6-D phase space volume element corresponding to a fixed particle mass m and a constant time hypersurface. The number of particles with mass m in the infinitesimal six dimensional phase space volume element at time t can be defined as

$$\hat{f}(x^i, t, p_i, m)d^3xd^3pdm, \qquad (2.32)$$

where $\hat{f}$ denotes the distribution function. The distribution $\hat{f}$ must be an (observer independent) invariant because, from equation (2.31), d^3xd^3pdm is also invariant. If the masses of the particles are restricted to a single value m_o, the distribution function can be written as

$$\hat{f}(x^i, t, p_i, m) = f(x^i, t, p_i, m_o)\delta_D(m - m_o). \tag{2.33}$$

The electromagnetic Hamiltonian can be calculated in the following manner. One takes the standard electromagnetic action

$$S_{EM} = -\frac{1}{4}\int (-g)^{1/2} d^4x F_{\mu\nu}F^{\mu\nu} \equiv$$
$$\int dt \int d^3x L \equiv \int dt \int d^3x (-g)^{1/2}\mathcal{L}, \tag{2.34}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and imposes the electromagnetic gauge condition $A_t = 0$. One can decompose the space-time metric $g_{\mu\nu}$, according to the ADM formalism, into the lapse function N, the shift vector N^i, and the spatial three-metric h_{ij}. The line element (cf. Arnowitt, Deser and Misner (1962, p. 227))

$$ds^2 = -g_{tt}dt^2 + 2g_{ti}dtdx^i + g_{ij}dx^i dx^j, \tag{2.35}$$

has metric components

$$g_{tt} = N_i N^i - N^2, \quad g_{it} = N_i$$

and

$$g_{ij} = h_{ij}. \tag{2.36}$$

The contravariant metric components are

$$g^{tt} = -\frac{1}{N^2}, \quad g^{it} = \frac{N^i}{N^2}$$

and

$$g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \tag{2.37}$$

One can raise and lower indices with the spatial three-metric. The inner product $h_{ij}h^{jk} = \delta_i{}^k$ furnishes an example of this. The "time" coordinate t is chosen so that an "equilibrium" corresponds to a space-time metric and electromagnetic field that have no t dependence. This "time" condition applies to static and stationary space-times, which possess timelike Killing vectors fields t^α, with an electromagnetic field seen as independent of time from a rest frame.

The field momentum can be calculated as

$$\begin{aligned}
\Pi^i &\equiv \frac{\partial L}{\partial(\partial_t A_i)} \\
= (-g)^{1/2} F^{it} &= \frac{h^{1/2}}{N} h^{ij} \partial_t A_j - h^{1/2} \frac{N^k}{N} (h^{ij} - \frac{N^i N^j}{N^2}) F_{kj} \\
&= \frac{h^{1/2}}{N} h^{ij} \partial_t A_j - h^{1/2} \frac{N^j}{N} h^{ik} F_{jk}
\end{aligned} \tag{2.38}$$

and inverted to yield the expression

$$\partial_t A_i = \frac{N}{h^{1/2}} h_{ij} \Pi^j + N^j F_{ji}. \tag{2.39}$$

The electromagnetic Hamiltonian

$$H_{EM}(\Pi^{it}, A_i) \equiv \int d^3x (\Pi^i \partial_t A_i - L) \tag{2.40}$$

can be written as

$$\begin{aligned}
H_{EM} &= \frac{1}{2} \int d^3x \frac{N}{h^{1/2}} h_{ij} \Pi^i \Pi^j + \int d^3x \Pi^i N^j F_{ji} \\
&+ \frac{1}{4} \int d^3x N h^{1/2} h^{ik} h^{jl} F_{ij} F_{kl}.
\end{aligned} \tag{2.41}$$

The matter Hamiltonian can be written as the minimal coupling between the distribution function f and the canonical particle energy ε

$$H_M = \int d\Gamma f \varepsilon. \tag{2.42}$$

The canonical particle energy $\varepsilon = -p_t$ satisfies the expression

$$g^{tt}{p_t}^2 + 2g^{ti}p_t(p_i - eA_i) + g^{ij}(p_i - eA_i)(p_j - eA_j) = -m^2, \tag{2.43}$$

where the gauge condition $A_t = 0$ has been imposed. The full electromagnetic Hamiltonian may be written as

$$H = H_{EM} + H_M - \int d^3x N h^{1/2} {j_B}^i A_i = H_{EM} + H_M + H_B, \tag{2.44}$$

where, in analogy with a fixed background charge density, an additional coupling term H_B, corresponding to a fixed, externally imposed charge current ${j_B}^i$, has been added. The vector potential A_i dependence of the energy ε, as made explicit in equation (2.43), implies that the matter Hamiltonian H_M couples the electromagnetic field to the Boltzmann distribution f. The equations (2.42) and (2.44) have been written in terms of a single species of matter. For the case of multiple species, we must define Boltzmann distributions f_a for each species a and replace (2.42) with a summation over separate distributions f_a.

The Lie bracket operation $\langle F, G\rangle$ will act on pairs of functionals $F[A_i, \Pi^i, f]$ and $G[A_i, \Pi^i, f]$ which are defined on the infinite dimensional phase space (A_i, Π^i, f). Consequently, the bracket may be written as

$$\langle F, G\rangle[A_i, \Pi^i, f] = \int d^3x \left(\frac{\delta F}{\delta A_i}\frac{\delta G}{\delta \Pi^i} - \frac{\delta G}{\delta A_i}\frac{\delta F}{\delta \Pi^i}\right) + \int d\Gamma f \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}, \tag{2.45}$$

where

$$\{a, b\} = \frac{\partial a}{\partial x^i}\frac{\partial b}{\partial p_i} - \frac{\partial b}{\partial x^i}\frac{\partial a}{\partial p_i}$$

represents the ordinary canonical Poisson bracket, and $\delta/\delta X$ represents the functional derivative corresponding to the variable X. It is straightforward to prove that the bracket is antisymmetric and satisfies the Jacobi identity

$$\langle F, \langle G, H\rangle\rangle + \langle G, \langle H, F\rangle\rangle + \langle H, \langle F, G\rangle\rangle = 0, \tag{2.46}$$

which implies that it is a *bona fide* Lie bracket. If one considers restricted functionals $F[f]$ and $G[f]$, that possess only f dependence, the bracket (2.45) reduces to

$$\langle F, G\rangle[f] = \int d\Gamma f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\}, \tag{2.47}$$

which is the proper bracket of the Vlasov-Poisson (Morrison (1980a, b); Gibbons (1981); Kandrup (1980)) system. For functionals $F[A_i, \Pi^i]$ and $G[A_i, \Pi^i]$, whose dependence is constrained to A_i and Π^i, the bracket becomes

$$\langle F, G\rangle[A_i, \Pi^i] = \int d^3x \left(\frac{\delta F}{\delta A_i} \frac{\delta G}{\delta \Pi^i} - \frac{\delta G}{\delta A_i} \frac{\delta F}{\delta \Pi^i} \right). \tag{2.48}$$

For vacuum electrodynamics (cf. Wald (1984)), formulated in terms of a specified 3+1 decomposition, this is the proper bracket.

One may obtain equation (2.48) from the covariant bracket

$$\langle\langle F, G\rangle\rangle = \int d^4x V^{\nu} \left(\frac{\delta F}{\delta A_{\mu}} \frac{\delta G}{\delta \Pi^{\mu\nu}} - \frac{\delta G}{\delta A_{\mu}} \frac{\delta F}{\delta \Pi^{\mu\nu}} \right), \tag{2.49}$$

where $\Pi^{\mu\nu} = \delta S / \delta(\partial_{\mu} A_{\nu})$ and V^{μ} is an arbitrary vector field corresponding to a suitable foliation. The electromagnetic bracket given by Marsden et al. (1986) reduces to equation (2.49) in the vacuum limit. For an appropriate choice of bracket and Hamiltonian, the constraint equation

$$\partial_t F = \langle F, H\rangle \tag{2.50}$$

should reproduce the correct dynamical equations of the Vlasov-Maxwell system. To prove that this is the case, one must first calculate the following three identities

$$\frac{\delta H}{\delta f} = \varepsilon, \tag{2.51a}$$

$$\frac{\delta H}{\delta \Pi^i} = \frac{N}{h^{1/2}} h_{ij} \Pi^j + N^j F_{ji}, \tag{2.51b}$$

and

$$\frac{\delta H}{\delta A_i} = \partial_j (N^j \Pi^i - N^i \Pi^j) + \partial_k (N h^{1/2} h^{ij} h^{kl} F_{lj}) - N h^{1/2} (J^i + {j_B}^i), \tag{2.51c}$$

where

$$J^i = e\int (-g)^{-1/2} d^3p f\left(\frac{p^i - eA^i}{P^t}\right).$$

One can use the three identities to prove that the Vlasov-Maxwell system is obtainable from equation (2.50). It is straightforward to check that the phase space variables, A_i, Π^i and f, satisfy the appropriate dynamical equations

$$\partial_t A_i = \langle A_i, H\rangle = \frac{\delta H}{\delta \Pi^i}, \tag{2.52a}$$

$$\partial_t \Pi^i = \langle \Pi^i, H\rangle = -\frac{\delta H}{\delta A_i} \tag{2.52b}$$

and

$$\partial_t f = \langle f, H\rangle = -\left\{f, \frac{\delta H}{\delta f}\right\} = \{\varepsilon, f\}. \tag{2.52c}$$

The first two of these equations are the canonical equations for an electromagnetic field with a total current source ${J_{tot}}^i = J^i + {j_B}^i$. The equation for $\partial_t A_i$ is equation (2.39) which defines the field momentum Π^i. Likewise, the second equation is the 3+1 decomposition of

$$(-g)^{1/2}\nabla_\mu F^{\mu i} = (-g)^{1/2} {J_{tot}}^i, \tag{2.53}$$

which is the Maxwell equation $-\partial_t \vec{E} + \vec{\nabla}\times\vec{B} = \vec{J}_{tot}$ generalized to curved space.

The Vlasov equation can be written as

$$\frac{\partial f}{\partial t} = -\frac{P^i}{P^t}\frac{\partial f}{\partial x^i} - e\frac{P^j}{P^t}\frac{\partial A_j}{\partial x^i}\frac{\partial f}{\partial p_i} + \frac{1}{2}\frac{P_\mu P_\nu}{P^t}\frac{\partial g^{\mu\nu}}{\partial x^i}\frac{\partial f}{\partial p_i}, \tag{2.54}$$

where $P_\mu \equiv p_\mu - eA_\mu$ and $P^\mu = g^{\mu\nu}(p_\nu - eA_\nu)$ are expressed in terms of the canonical variables x^i and p_i. This Vlasov equation is different from the usual one which is expressed in terms of the canonical momentum, with $f(x^i, t, p_i)$. However, a transformation to the physical coordinates (x^i, t, P_i) will demonstrate that $f(x^i, t, P_i)$ satisfies the standard Vlasov equation (cf. eq. (2.29))

$$\frac{\partial f}{\partial t} - e\frac{\partial A_i}{\partial t}\frac{\partial f}{\partial P_i} = -\frac{P^i}{P^t}\left[\frac{\partial f}{\partial x^i} - e\frac{\partial A_j}{\partial x^i}\frac{\partial f}{\partial P_j}\right]$$

$$+e\frac{P^j}{P^t}\frac{\partial A_j}{\partial x^i}\frac{\partial f}{\partial p_i}+\frac{1}{2}\frac{P_\mu P_\nu}{P^t}\frac{\partial g^{\mu\nu}}{\partial x^i}\frac{\partial f}{\partial p_i}. \tag{2.55}$$

One of the curved space dynamical Maxwell equations is given by equation (2.52b). One may obtain the curved space generalization of the other dynamical Maxwell equation $\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0$. For example, we may define the quantity

$$B^i = \epsilon^{ijk}\partial_j A_k = h^{1/2} e^{ijk}\partial_j A_k. \tag{2.56}$$

Through the use of the alternating symbol ϵ^{ijk} or the alternating tensor e^{ijk}, it will satisfy the following equation of motion

$$\partial_t B^i = \epsilon^{ijk}\partial_j\partial_t A_k = \partial_j\left[\epsilon^{ijk}\left(\frac{N}{h^{1/2}}h_{kl}\Pi^l + N^l F_{kl}\right)\right]. \tag{2.57}$$

In an alternate formulation, $\partial_t B^i$ can be written in terms of the electric $E^i \equiv -\Pi^i$ and magnetic B^i field densities

$$\partial_t B^i = -\partial_j\left(\frac{N}{h^{1/2}}\epsilon^{ijk}h_{kl}E^l\right) + \partial_j\left(B^i N^j - B^j N^i\right). \tag{2.58}$$

Either equation (2.57) or (2.58) is the dynamical component of the dual Maxwell equation $\nabla_\mu {}^*F^{\mu\nu} = 0$. One can construct the curved space generalizations of the constraint equations $\nabla \cdot \vec{B} = 0$ and $\nabla \cdot \vec{E} = \rho$. The first is a geometric identity resulting from the fact that B^i is constructed from an alternating symbol. The second constraint possesses a more complex character. The continuity equation $\nabla_\mu J^\mu = 0$ and the dynamical equation for $\nabla_\mu F^{\mu i} = 0$ ensure that, if the electric field constraint is initially enforced, the constraint will be maintained through time. Also, gauge invariance (cf. Misner, Thorne and Wheeler (1973)) can be used to demonstrate that the electric constraint will hold initially. It is possible to rewrite the bracket (2.45) in terms of variables which are physically more intuitive. According to Marsden and Weinstein (1982), the bracket may be written in terms of $E^i =$

$-\Pi^i$ and B^i. These variables correspond to the true electromagnetic degrees of freedom and appear in equation (2.58). The bracket will take the form

$$\begin{aligned}\langle \hat{F}, \hat{G}\rangle = \int d^3x h^{1/2} e^{ijk}\left[\frac{\delta \hat{F}}{\delta E^i}\frac{\partial}{\partial x^j}\left(\frac{\delta \hat{G}}{\delta B^k}\right) - \frac{\delta \hat{G}}{\delta E^i}\frac{\partial}{\partial x^j}\left(\frac{\delta \hat{F}}{\delta B^k}\right)\right] \\ + \int d\Gamma f\left\{\frac{\delta \hat{F}}{\delta f}, \frac{\delta \hat{G}}{\delta f}\right\},\end{aligned} \tag{2.59}$$

with functionals $\hat{F}[B^i, E^i, f]$ and $\hat{G}[B^i, E^i, f]$. Also, if one rewrites the distribution $\tilde{f}(x^i, t, P_i)$ in terms of the physical momentum, instead of the canonical p_i, this will lead to a modified bracket form

$$\begin{aligned}\langle \bar{F}, \bar{G}\rangle = \int d^3x h^{1/2} e^{ijk}\left[\frac{\delta \bar{F}}{\delta E^i}\frac{\partial}{\partial x^j}\left(\frac{\delta \bar{G}}{\delta B^k}\right) - \frac{\delta \bar{G}}{\delta E^i}\frac{\partial}{\partial x^j}\left(\frac{\delta \bar{F}}{\delta B^k}\right)\right] \\ + \int d\Gamma \tilde{f}\left\{\frac{\delta \bar{F}}{\delta \tilde{f}}, \frac{\delta \bar{G}}{\delta \tilde{f}}\right\} \\ + \int d\Gamma\left(\frac{\delta \bar{F}}{\delta E^i}\frac{\partial \tilde{f}}{\partial p_i}\frac{\delta \bar{G}}{\delta \tilde{f}} - \frac{\delta \bar{G}}{\delta E^i}\frac{\partial \tilde{f}}{\partial p_i}\frac{\delta \bar{F}}{\delta \tilde{f}}\right) \\ + \int d\Gamma h^{-1/2} e_{ijk} B^i \frac{\partial}{\partial p_j}\left(\frac{\delta \bar{F}}{\delta \tilde{f}}\right)\frac{\partial}{\partial p_k}\left(\frac{\delta \bar{G}}{\partial \tilde{f}}\right)\tilde{f},\end{aligned} \tag{2.60}$$

with functionals $\bar{F}[B^i, E^i, \tilde{f}]$ and $\bar{G}[B^i, E^i, \tilde{f}]$. It is clear that equations (2.59) and (2.60) correspond to covariant analogues of the equations (5.1) and (7.1) found in Kandrup and Morrison (1993).

2.6 Perturbations of Time-Independent Vlasov-Maxwell Equilibria

If there is to be a meaningful definition of "equilibrium" then the curved space, which the Vlasov-Maxwell system is situated in, will be a static or stationary space-time that possesses a timelike Killing vector. An "equilibrium" will correspond to the case where both the electromagnetic field and matter distribution are independent of time. Consequently, a "natural" choice of time coordinate t will cause the metric $g_{\mu\nu}$ and electromagnetic field $F_{\mu\nu}$ to lack time dependence. This choice leads to a time derivative of the canonical momentum

Π^i that is equal to zero. The "electric field" Π^t must vanish if the derivative $\partial_t A_i$ is to be independent of time. However, unless the second derivative $\partial_t{}^2 A_i \equiv 0$, the $F_{\mu\nu}$ tensor will not be independent of time. When the canonical momentum $p_i = P_i + eA_i$ is expressed in terms of the physical variables x^i and P_i, it will, at "equilibrium", possess explicit time dependence because of the non-vanishing of $\partial_t A_i$. Therefore, for equilibria of this type, the time derivative $\partial_t f_o$ cannot be set to zero. Instead, the equilibrium distribution f_o must obey the relation

$$\frac{\partial f_o}{\partial t} = e\frac{\partial A_i}{\partial t}\frac{\partial f_o}{\partial P_i} = e\frac{\partial A_i}{\partial t}\frac{\partial f_o}{\partial p_i}. \tag{2.61}$$

One wishes to know, for a given equilibrium $X_o = (A_i{}^o, \Pi^i{}_o, f_o)$, what mathematical form the dynamically accessible perturbation $\delta X = (\delta A_i, \delta\Pi^i, \delta f)$ will possess. It is straightforward to determine this because there exists an infinite number of conserved constraints, corresponding to conservation of phase, that are associated with evolution in the Vlasov-Maxwell system. Specifically, this implies that the numerical value of any functional constraint

$$C[f(t)] = \int d\Gamma \chi(f) \tag{2.62}$$

remains constant as the distribution f evolves, i.e.,

$$dC[f]/dt \equiv 0. \tag{2.63}$$

Consequently, any dynamically accessible perturbation δf must leave the numerical value of any such constraint unchanged. It follows that

$$\delta C[f] \equiv C[f_o + \delta f] - C[f_o] \equiv 0, \tag{2.64}$$

for all orders of perturbation theory. A perturbation must obey this mathematical requirement to be dynamically accessible, i.e., in accord with the equations of dynamics. A

solution to (2.64) will yield the form of the phase-preserving perturbation. It may be shown perturbatively that the deformation δf, which serves as a solution (Bloch, Krishnaprasad, Marsden and Ratiu (1991)) to equation (2.64), must be

$$f = f_o + \delta f = \exp(\{g, .\}) f_o = f_o + \{g, f_o\} + \frac{1}{2}\{g, \{g, f_o\}\} + ..., \tag{2.65}$$

where g is a generating function of the variables x^i and p_i, and $\{a, b\}$ denotes the ordinary canonical Poisson bracket. So, a canonical transformation, with a generating function g, corresponds to a dynamical perturbation. Unlike the distribution f_o, the canonical variables A_i and Π^i do not satisfy corresponding independent constraint equations similar to equation (2.61). Thus, the dynamically accessible perturbations δA_i and $\delta \Pi^i$ are only constrained by the field equations whose source is provided by the dynamically accessible perturbation δf.

One can demonstrate that the first variation $\delta^{(1)} H$ of the Hamiltonian (2.44) vanishes for a dynamically accessible perturbation $\delta^{(1)} f$ (2.65). The first variation of the electromagnetic part of the Hamiltonian possesses the form

$$\begin{aligned} \delta^{(1)} H_{EM} &= \int d^3x \delta \Pi^i (\frac{N}{h^{1/2}} h_{ij} \Pi^j + N^j F_{ji}) - \int d^3x \delta A_i \partial_j (\Pi^i N^j - \Pi^j N^i) \\ &\quad - \int d^3x \delta A_i \partial_j (N h^{1/2} h^{ik} h^{jl} F_{kl}) \\ &= \int d^3x \delta \Pi^i (\partial_t A_i) + \int d^3x N h^{1/2} \delta A_i \nabla_\mu F^{\mu i}. \end{aligned} \tag{2.66}$$

The last equality has been obtained from equation (2.39) for $\partial_t A_i$ and the fact that, for the case of a time-independent equilibrium with $\partial_t \Pi^i = 0$, terms proportional to δA_i may be combined to form $\nabla_\mu F^{\mu i}$. Utilizing the unperturbed Maxwell equation $\nabla_\mu F^{\mu\nu} = J^\nu + {j_B}^\nu$ leads to the expression

$$\delta^{(1)} H_{EM} = \int d^3x (\partial_t A_i) \delta \Pi^i + \int d^3x N h^{1/2} (J^i + {j_B}^i) \delta A_i. \tag{2.67}$$

Also, the matter Hamiltonian has the first variation

$$\delta^{(1)}H_M = \int d\Gamma\left(f\frac{\partial\varepsilon}{\partial A_i}\delta A_i + \varepsilon\delta f\right) = -\int d^3xNh^{1/2}J^i\delta A_i$$
$$+\int d\Gamma\varepsilon\delta f. \tag{2.68}$$

The coupling Hamiltonian will have the variational form

$$\delta^{(1)}H_B = -\int d^3xNh^{1/2}{j_B}^i\delta A_i. \tag{2.69}$$

One may combine equations (2.67)-(2.69) to get the total variation of the Hamiltonian

$$\delta^{(1)}H = \int d^3x(\partial_t A_i)\delta\Pi^i + \int d\Gamma\varepsilon\delta f. \tag{2.70}$$

This last expression has been, of course, calculated for an arbitrary perturbation. With the special choice of a dynamically accessible perturbation, it is straightforward to show that this variation is vanishing. For an equilibrium, the magnetic field is static $\partial_t B^i = 0$. From equation (2.57) one can deduce that $\partial_t A_i$ may be written as the gradient of an arbitrary scalar, so that

$$\partial_t A_i = -\partial_i\Psi. \tag{2.71}$$

Substitution of equation (2.71) into the first term of equation (2.70) yields the expression

$$\int d^3x(\partial_t A_i)\delta\Pi^i = -\int d^3xNh^{1/2}\partial_i\Psi\delta F^{it} = \int d^3x\Psi\partial_i[Nh^{1/2}\delta F^{it}]$$
$$= \int d^3xNh^{1/2}\Psi\nabla_i\delta F^{it}, \tag{2.72}$$

where an integration by parts has been performed and the expression $\Pi^i = Nh^{1/2}F^{it}$ has been employed. Using the perturbed field equation and writing δJ^t in terms of δf leads to

$$\int d^3x(\partial_t A_i)\delta\Pi^i = \int d^3xNh^{1/2}\Psi\delta J^t = e\int d\Gamma\Psi\delta f. \tag{2.73}$$

By substitution of the dynamically accessible form of δf (2.65) into this expression, one can obtain the final result

$$e\int d\Gamma\Psi\delta f = e\int d\Gamma\Psi\{g, f_o\} = -e\int d\Gamma g\{\Psi, f_o\}. \tag{2.74}$$

Also, substitution of the dynamically accessible δf into the second term of equation (2.70) leads to

$$\begin{aligned}\int d\Gamma\varepsilon\delta f = \int d\Gamma\varepsilon\{g, f_o\} = -\int d\Gamma g\{\varepsilon, f_o\} = -\int d\Gamma g\partial_t f_o = \\ -e\int d\Gamma g\partial_t A_i\frac{\partial f_o}{\partial p_i}.\end{aligned} \tag{2.75}$$

The final expression of (2.75) may be rewritten in the following form

$$e\int d\Gamma g(\partial_i\Psi)\frac{\partial f_o}{\partial p_i} = e\int d\Gamma g\{\Psi, f_o\}, \tag{2.76}$$

by replacing $\partial_t A_i$ with $-\partial_i\Psi$. Consequently, both the first (cf. 2.74) and second terms cancel out with each other. Therefore, the energy H has an extremal $\delta^{(1)}H = 0$ for a time-independent equilibrium in the curved space Vlasov-Maxwell system. In this setting, the space-time possesses a timelike Killing vector field and, at equilibrium, a static electromagnetic field $F_{\mu\nu}$ (the electric and magnetic fields are time-independent as viewed by a fixed observer). One now may wish to calculate the second variation $\delta^{(2)}H$. This variation may be conveniently reduced to the sum of three terms

$$\delta^{(2)}H = \delta^{(2)}H_1 + \delta^{(2)}H_2 + \delta^{(2)}H_3, \tag{2.77}$$

where

$$\begin{aligned}\delta^{(2)}H_1 = \frac{1}{2}\int d^3x\frac{N}{h^{1/2}}h_{ij}\delta\Pi^i\delta\Pi^j + \int d^3x\delta\Pi^i N^j\delta F_{ji} \\ +\frac{1}{4}\int d^3x N h^{1/2}h^{ik}h^{jl}\delta F_{ij}\delta F_{kl}\end{aligned} \tag{2.78}$$

are the terms from H_{EM} that possess products of first variations,

$$\delta^{(2)}H_2 = \int d^3x\delta^{(2)}\Pi^i(\frac{N}{h^{1/2}}h_{ij}\Pi^j + N^j F_{ji})$$

$$-\int d^3x\delta^{(2)}A_i\partial_j(\Pi^iN^j-\Pi^jN^i)-\int d^3x\delta^{(2)}A_i\partial_j(Nh^{1/2}h^{ik}h^{jl}F_{kl}) \tag{2.79}$$

are the terms from H_{EM} that have second variations of canonical variables, and

$$\delta^{(2)}H_3=\int d\Gamma[\varepsilon\delta^{(2)}f+2\delta\varepsilon\delta f+\delta^{(2)}\varepsilon f_o]$$

$$-\int d^3xNh^{1/2}{j_B}^i\delta^{(2)}A_i \tag{2.80}$$

are terms corresponding to H_M and H_B. In these expressions, ε represents the unperturbed particle energy and a variation δX not possessing a superscript represents a first variation $\delta^{(1)}$. One can readily calculate that the first and second variations of the particle energy ε are

$$\delta\varepsilon=\frac{\partial\varepsilon}{\partial A_i}\delta A_i=-e\left(\frac{P^i-eA^i}{P^t}\right)\delta A_i \tag{2.81}$$

and

$$\begin{aligned}\delta^{(2)}\varepsilon&=\frac{1}{2}\frac{\partial^2\varepsilon}{\partial A_i\partial A_j}\delta A_i\delta A_j+\frac{\partial\varepsilon}{\partial A_i}\delta^{(2)}A_i\\&=-\frac{e}{2}\delta\left(\frac{P^i-eA^i}{P^t}\right)\delta A_i-e\left(\frac{P^i-eA^i}{P^t}\right)\delta^{(2)}A_i.\end{aligned} \tag{2.82}$$

Therefore, combining the second and third terms in $\delta^{(2)}H_3$ leads to the result

$$\int d\Gamma[\delta^{(2)}\varepsilon f_o+\delta\varepsilon\delta f]=$$

$$\begin{aligned}&-\int Nh^{1/2}d^3xJ^i\delta^{(2)}A_i-e\int d\Gamma\left[\delta\left(\frac{P^i-eA^i}{P^t}\right)f_o+\left(\frac{P^i-eA^i}{P^t}\right)\delta f\right]\delta A_i\\&+\frac{e}{2}\int d\Gamma f_o\delta\left(\frac{P^i-eA^i}{P^t}\right)\delta A_i=-\int d^3xNh^{1/2}J^i\delta^{(2)}A_i-\int d^3x\delta J^i\delta A_i\\&-e^2\frac{1}{2}\int Nh^{1/2}d^3xNh^{1/2}\Theta^{ij}\delta A_i\delta A_j,\end{aligned} \tag{2.83}$$

where

$$\Theta^{ij}=\int(-g)^{-1/2}d^3pf_o\frac{\partial^2\varepsilon}{\partial A_i\partial A_j}.$$

Using the unperturbed field equation allows one to rewrite the second contribution $\delta^{(2)}H_2$ as

$$\delta^{(2)}H_2 = \int d^3x\delta^{(2)}\Pi^i(\partial_t A_i) + \int d^3x(-g)^{1/2}\delta^{(2)}A_i\nabla_\mu F^{\mu i}$$
$$= \int d^3x\delta^{(2)}\Pi^i(\partial_t A_i) + \int d^3xNh^{1/2}\delta^{(2)}A_i(J^i + j_B{}^i). \tag{2.84}$$

One can combine the second term of equation (2.84) with the last term of equation (2.80) to cancel out the first term of equation (2.83). Therefore, in the case of an arbitrary perturbation δX, the second variation $\delta^{(2)}H$ will be

$$\delta^{(2)}H = \frac{1}{2}\int d^3x\frac{N}{h^{1/2}}h_{ij}\delta\Pi^i\delta\Pi^j + \int d^3x\delta\Pi^iN^j\delta F_{ji}$$
$$+\frac{1}{4}\int d^3xNh^{1/2}h^{ik}h^{jl}\delta F_{ij}\delta F_{kl} - \frac{e^2}{2}\int d^3xNh^{1/2}\Theta^{ij}\delta A_i\delta A_j$$
$$-\int d^3xNh^{1/2}\delta J^i\delta A_i$$
$$+\int d^3x\delta^{(2)}\Pi^i(\partial_t A_i) + \int d\Gamma\varepsilon\delta^{(2)}f. \tag{2.85}$$

Restricting attention to perturbations that are dynamically accessible allows one to rewrite the last two terms of equation (2.85) in a simpler form. Using the scalar function Ψ, one may write

$$\int d^3x\delta^{(2)}\Pi^i\partial_t A_i = \int d^3x\Psi\partial_i\delta^{(2)}\Pi^i$$
$$= \int d^3xNh^{1/2}\Psi\nabla_i\delta^{(2)}F^{it} = \int Nh^{1/2}d^3x\Psi\delta^{(2)}J^t. \tag{2.86}$$

In the case of a dynamically accessible perturbation, one can rewrite equation (2.86) as

$$e\int d\Gamma\Psi\delta^{(2)}f = e\int d\Gamma\frac{1}{2}\{g,\{g,f_o\}\}\Psi = -\frac{1}{2}\int d\Gamma\{g,f_o\}\{g,e\Psi\}. \tag{2.87}$$

Likewise, the last term of equation (2.85) will take the form

$$\int d\Gamma\varepsilon\delta^{(2)}f = \frac{1}{2}\int d\Gamma\{g,\{g,f_o\}\}\varepsilon = -\frac{1}{2}\int d\Gamma\{g,f_o\}\{g,\varepsilon\}. \tag{2.88}$$

Therefore, for a dynamically accessible perturbation, the second order variation $\delta^{(2)}H$ possesses the form

$$\delta^{(2)}H = \frac{1}{2}\int d^3x \frac{N}{h^{1/2}} h_{ij}\delta\Pi^i\delta\Pi^j + \int d^3x N^j \delta\Pi^i \delta F_{ji}$$
$$+\frac{1}{4}\int d^3x N h^{1/2} h^{ik}h^{jl}\delta F_{ij}\delta F_{kl} - \frac{e^2}{2}\int d^3x N h^{1/2}\Theta^{ij}\delta A_i \delta A_j$$
$$-\int d^3x N h^{1/2}\delta J^i \delta A_i - \frac{1}{2}\int d\Gamma\{g, f_o\}\{g, \varepsilon + e\Psi\}. \tag{2.89}$$

This expression can be reformulated in terms of the electric and magnetic field densities (E^i and B^i) as

$$\delta^{(2)}H = \frac{1}{2}\int d^3x \frac{N}{h^{1/2}} h_{ij}(\delta E^i \delta E^j + \delta B^i \delta B^j) - \int d^3x \frac{N^i}{h^{1/2}} e_{ijk}\delta E^j \delta B^k$$
$$-\frac{e^2}{2}\int d^3x N h^{1/2}\Theta^{ij}\delta A_i \delta A_j - \int d^3x N h^{1/2}\delta J^i \delta A_i$$
$$-\frac{1}{2}\int d\Gamma\{g, f_o\}\{g, \varepsilon + e\Psi\}, \tag{2.90}$$

where, as before, the alternating tensor is denoted as e_{ijk}.

2.7 Stability Criteria for Time-Independent Vlasov-Maxwell Equilibria

It is straightforward to show that the Poisson bracket $\{g, \varepsilon + e\Psi\}$, which appears in equation (2.90), possesses the mathematical form

$$\{g, \varepsilon + e\Psi\} = \{g, E\} = \frac{P^i}{P^t}\frac{\partial g}{\partial x^i} + e\frac{\partial \Psi}{\partial x^i}\frac{\partial g}{\partial p_i}$$
$$+e\frac{P^j}{P^t}\frac{\partial A_j}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{1}{2}\frac{P_\mu P_\nu}{P^t}\frac{\partial g^{\mu\nu}}{\partial x^i}\frac{\partial g}{\partial p_i}. \tag{2.91}$$

If the physical momenta P_i are considered fundamental, it may be rewritten as

$$\{g, E\} = \frac{P^i}{P^t}\frac{\partial g}{\partial x^i} + e\frac{\partial \Psi}{\partial x^i}\frac{\partial g}{\partial P_i} - eF_{ji}\frac{P^j}{P^t}\frac{\partial g}{\partial P_i} - \frac{1}{2}\frac{P_\mu P_\nu}{P^t}\frac{\partial g^{\mu\nu}}{\partial x^i}\frac{\partial g}{\partial P_i}. \tag{2.92}$$

The definition of Ψ leads to the result

$$\{g, E\} = -\mathcal{D}g, \tag{2.93}$$

where

$$\mathcal{D} \equiv \frac{P^i}{P^t}\frac{\partial}{\partial x^i} - e\frac{\partial A_i}{\partial t}\frac{\partial}{\partial P_i} + e\frac{P^j}{P_t}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\frac{\partial}{\partial P_i} + \frac{1}{2}\frac{P_\mu P_\nu}{P^t}\frac{\partial g^{\mu\nu}}{\partial x^i}\frac{\partial}{\partial P_i}$$

represents the unperturbed Liouville operator. The evolution of the distribution occurs along the characteristics corresponding to time-independent equilibria. This evolution is governed by the Liouville operator. The evolution equations do not imply conservation of the canonical energy ε whether there exists a time-independent equilibrium or not. Contrary to expectation, one will find

$$\frac{d\varepsilon}{dt} = \frac{\partial \varepsilon}{\partial t} = -e\frac{P^i}{P^t}\frac{\partial A_i}{\partial t}, \tag{2.94}$$

which, due to the time dependence of the vector potential, will not vanish. Since Π is time-independent, one may compute

$$\frac{\partial \Psi}{\partial t} = \frac{P^i}{P^t}\frac{\partial A_i}{\partial t}, \tag{2.95}$$

which implies that $dE/dt \equiv 0$. It follows that E, not ε, will be a conserved quantity for a time-independent equilibrium.

One can construct a time-independent solution $f_o(E, m)$ to the Vlasov equation from these constants of the motion. Other equilibria are possible, however, this choice corresponds to an ensemble of particles with an isotropic velocity distribution. Such equilibria are ubiquitous. For this choice, equation (2.90) may be written as

$$\begin{aligned}\delta^{(2)}H &= \frac{1}{2}\int d^3x \frac{N}{h^{1/2}} h_{ij}(\delta\Pi^i\delta\Pi^j + \delta B^i \delta B^j) \\ &+ \int d^3x \frac{N^i}{h^{1/2}} e_{ijk}\delta\Pi^j\delta B^k - \frac{e^2}{2}\int d^3x N h^{1/2}\Theta^{ij}\delta A_i \delta A_j \\ &- \int d^3x N h^{1/2}\delta J^i \delta A_i + \frac{1}{2}\int d\Gamma(-F_E)|\{g, E\}|^2. \end{aligned}\tag{2.96}$$

One may examine the various limits of (2.96) or (2.90). In the case of pure vacuum electromagnetism, the energy associated with a linearized vacuum fluctuation possesses the

form

$$\delta^{(2)}H = \frac{1}{2}\int d^3x \frac{N}{h^{1/2}} h_{ij}(\delta\Pi^i\delta\Pi^j + \delta B^i \delta B^j)$$
$$+ \int d^3x \frac{N^i}{h^{1/2}} e_{ijk}\delta\Pi^j\delta B^k, \tag{2.97}$$

which is obtained from equation (2.96) by turning off the matter distribution. As another example, one may examine spherically symmetric configurations in the non-relativistic approximation. Spherical symmetry implies that the shift vector N^i vanishes and the slow motion approximation implies that the contributions $\delta J^i \delta A_i$ and $\delta A_i \delta A_j$, which are of order $(v/c)^2$, may be safely neglected. This yields the covariant analogue of the linearized energy associated with the flat space Vlasov-Poisson system, viz.

$$\delta^{(2)}H = \frac{1}{2}\int d^3x \frac{N}{h^{1/2}} h_{ij}\delta\Pi^i\delta\Pi^j - \frac{1}{2}\int d\Gamma \{g, f_o\}\{g, E\}. \tag{2.98}$$

One can deduce nontrivial stability criteria from the sign of the energy $\delta^{(2)}H$. If, for example, $\delta^{(2)}H \geq 0$, for all generating functions g, then any static equilibrium must be linearly stable. On the other hand, if, for some generating function g, the condition $\delta^{(2)}H < 0$ holds, then a linear instability is not necessarily guaranteed. Such a configuration, although it possesses linear stability, will probably be nonlinearly unstable or unstable if subjected to dissipative effects (Holm, Marsden, Ratiu and Weinstein (1985); Moser (1968); Morrison (1987)). Whether an equilibrium corresponds to an energy minimum is, generally speaking, difficult to determine. There is, however, one case where it is simple to show that the condition $\delta^{(2)}H < 0$ exists for some phase-preserving perturbation. For example, assume that the energy derivative F_E is not everywhere negative. If one chooses a perturbation such that

$$\delta J^\mu = \int d\Gamma \{g, f_o\} \frac{P^\mu}{P^t} = 0, \tag{2.99}$$

i.e., the overall charge density remains unaffected but the velocity profile is shuffled, then

one can assume δA_i and $\delta\Pi^i$ are zero. However, the dynamical perturbation must obey the relation

$$\delta^{(2)}H = \frac{1}{2}\int d\Gamma(-F_E)|\{g,E\}|^2. \tag{2.100}$$

By shuffling the velocities a significant amount where F_E is positive in phase space, and by a minimal amount elsewhere, one can obtain a negative energy (2.100). This is the curved space analogue of the fact, first observed by Morrison and Pfirsch (1989), that "all interesting equilibria are either linearly unstable or possess negative energy modes."

CHAPTER 3
THE VLASOV-EINSTEIN SYSTEM

One may now consider the full covariant Vlasov-Einstein system. We attempt to give a Hamiltonian formulation of the collisionless Boltzmann equation of General Relativity (Vlasov-Einstein system). Energy stability criteria will be deduced for this Hamiltonian system. For this system (cf. Israel (1972, p. 201); Stewart (1971)), the Boltzmann distribution f will be defined with respect to the cotangent bundle corresponding to the space-time of choice. The characteristics, which correspond to geodesics of the space-time, are the trajectories along which the distribution f evolves. The system is complicated by nonlinearity because the space-time is determined by the distribution f which acts as a source of the mean field Einstein equation. Thus, the background space-time is not fixed and eternal as it was for the case of the Vlasov-Maxwell system in chapter 2. Therefore, in this scenario, the Boltzmann distribution has its evolution governed by the Vlasov equation

$$\frac{p^a}{m}\frac{\partial f}{\partial x^a} - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}\frac{p_b p_c}{m}\frac{\partial f}{\partial p_a} = 0, \quad (a, b, c, ... = 0, 1, 2, 3) \tag{3.1}$$

where the space-time metric g_{ab} must self-consistently satisfy the mean field Einstein equation

$$G^a{}_b = 8\pi T^a{}_b = 8\pi \int \frac{d^4p}{(-g)^{1/2}}\frac{f}{m}p^a p_b. \tag{3.2}$$

There have been earlier investigations of the linear stability of restricted cases of the Vlasov-Einstein system. One example corresponds to spherically symmetric equilibrium solutions whose perturbations are also spherically symmetric. For example, in the earliest investigation by Ipser and Thorne (1968), a variational principle is derived from the dy-

namical equations. Unstable modes can be detected by analyzing the variational principle. Ipser and Thorne (1968) write the perturbed distribution δf as a sum $\delta f = \delta f_- + \delta f_+$, where δf_+ and δf_- are even and odd, respectively, with respect to spatial momentum inversion. It was noted that, in certain circumstances, the odd perturbation component δf_- satisfies the operator equation $\partial_t{}^2 \delta f_- = \mathcal{T} \delta f_-$. This equation is second order in time and the symmetric operator $\mathcal{T}$, which is defined with respect to a suitable Hilbert space, is dependent only on the equilibrium f_o. The pioneering work on this technique was carried out by Antonov (1961), albeit in a non-relativistic setting. When applied to a relativistic system, this method has been subjected to extensive numerical analysis by Ipser (1969a, b) and Fackerell (1970, 1971) among others.

Ipser (1980) then went on to analyze spherically symmetric systems in terms of plasma physics techniques. An extension of the Newtonian "energy arguments" (cf. Ipser and Horwitz (1979)) was carried out with the use of an energy-Casimir argument to obtain linear stability criteria. This approach, due to Newcomb (cf. the summary in the appendix of Bernstein (1958)), had originally appeared in plasma physics. This approach is also discussed by Morrison and Pfirsch (1989) (and in internally cited references). The basic idea is as follows. For generic perturbations δf, equilibrium solutions f_o of the Vlasov-Einstein system do not correspond to energy extremals because the constraints associated with phase conservation imply the existence of infinitely many conserved quantities, $C[f]$. It is possible, however, for equilibria to be extremal by considering only perturbations which ensure the conservation of one particular constraint, $c[f]$. Thus, the sign of the constrained second variation, $\delta^{(2)} H_c$, yields a nontrivial stability criterion. For example, the condition $\delta^{(2)} H_c \geq 0$ ensures linear stability.

Recent work has applied these plasma physics arguments (cf. the papers of Morrison and Greene (1980), Morrison (1980a) and Morrison and Pfirsch (1989), and internally cited references) to Newtonian galactic dynamics (cf. Kandrup (1990, 1991a, b)). Kandrup and Morrison (1993) have adapted these ideas in order to furnish a more systematic Hamiltonian formulation. In this work, the Hamiltonian character of the dynamical evolution, which was ignored in earlier research, is addressed. Also, all of the constraints (not just one) are implemented. Consequently, the earlier work of Ipser (1980) is clarified and extended. So far, only the stability of spherically symmetric equilibria, subjected to spherically symmetric perturbations, have been studied. Research prior to Kandrup and Morrison (1993) assumed that the Boltzmann distribution was restricted in form. For example, the distribution was supposed to be a monotonically decreasing function of the particle energy ε.

Lie algebraic techniques were employed by Kandrup and Morrison (1993) to show that the spherically symmetric Vlasov-Einstein system, subjected to a three-plus-one decomposition into space and time, is Hamiltonian. Consequently, it is necessary to identify a bracket $[.,.]$, i.e., a cosymplectic structure and a Hamiltonian function H, which are used to construct a Vlasov equation $\partial_t f = [f, H]$ for the distribution function f, where ∂_t denotes a coordinate time derivative. Since the dynamical evolution is governed by a "generalized canonical transformation" in the infinite-dimensional phase space associated with distribution functions f, it is obvious that the Vlasov-Einstein system is Hamiltonian in character. However, the transformation is not truly canonical since f is a non-canonical variable.

Due to constraints that are associated with conservation of phase, the Hamiltonian system is constrained. These phase-preserving constraints can be straightforwardly implemented in an explicit fashion. This enables one to prove that time-independent equilibria are energy extremals, such that $\delta^{(1)}H = 0$ for all dynamically accessible perturbations δf.

If the second variation $\delta^{(2)}H$ is non-negative for all dynamically accessible perturbations, then the system is guaranteed to be linearly stable. However, as noted in Kandrup and Morrison (1993), a negative second variation $\delta^{(2)}H < 0$, corresponding to some dynamically accessible perturbation δf, does not prove linear instability. It should, however, at least guarantee structural instability with regard to dissipation or any nonlinearities which are present (cf. Moser (1968); Morrison (1987); Bloch et al. (1991)).

Unfortunately, although this approach has proven highly effective, it is still restricted to systems, whether perturbed or not, that are spherically symmetric. Most related research, with the notable exception of Ipser and Semenzato (1979), has been directly based on this assumption. Due to the Birkhoff theorem, the spherical symmetry of the system ensures that gravitational radiation will not be present, i.e., the radiative degrees of freedom will not be activated by the dynamics. Therefore, for appropriate boundary conditions at infinity or any horizons that may be present, the gravitational metric g_{ab} at time t will be uniquely determined by the stress-energy tensor T_{ab} at that instant. This can be seen by implementing the standard Schwarzschild gauge, where only the pure radial g_{rr} and pure temporal g_{tt} metric components remain free. For this particular gauge, the $t-t$ component of the Einstein field equation fixes g_{rr} uniquely in terms of the energy density $T^t{}_t$. Also, if g_{rr} and the radial "pressure" $T^r{}_r$ are both known, then the $r-r$ component of the field equation will fix g_{tt}.

This situation is analogous to the externally imposed curved space Vlasov-Maxwell system considered in chapter 2. For the case of spherical symmetry, the magnetic divergence equation guarantees the vanishing of the magnetic field. The symmetry ensures that only the radial component of the electric field E_r does not vanish. The radial component is fixed uniquely by solving the Poisson equation for a charge distribution ρ. It is possible to

devise a Hamiltonian formulation which is applicable to general space-times, rather than only the spherically symmetric case considered by Kandrup and Morrison (1993). This will complicate the physics because the gravitational metric g_{ab} must be treated as a dynamical variable, as well as the Boltzmann distribution f_o. Kandrup and Morrison employed only the distribution function f_o as a dynamical variable. They worked in the context of an infinite-dimensional phase space of distribution functions f_o at a given instant of time t. For a general space-time, we must employ three dynamical variables. These are (1) the spatial three-metric $h_{ab}(x^i)$ for a $t =$ constant hypersurface and (2) the conjugate momentum $\Pi^{ab}(x^i)$, along with (3) the Boltzmann distribution f_o. One can define functionals $F[h_{ab}, \Pi^{ab}, f]$ in terms of the infinite-dimensional phase space (h_{ab}, Π^{ab}, f). It is necessary to select a Lie bracket $\langle ., . \rangle$ and a Hamiltonian function H such that the correct dynamical equations governing the evolution of h_{ab}, Π^{ab}, and f are generated by the constraint equation $\partial_t F = \langle F, H \rangle$.

3.1 The Vlasov-Einstein Hamiltonian Formulation

It will be convenient for the dynamical analysis of a combined space-time-distribution system to invoke an ADM decomposition into space and time (cf. Wald (1984)). One may view the space-time, described by a metric g_{ab}, as a foliation into a series of spacelike hypersurfaces. The hypersurfaces are parametrized by a global time function t, and n^a will denote a unit normal vector to the hypersurface. The spatial three-metric $h_{ab}(x^i)$ for a $t =$ constant hypersurface is defined by the projection tensor,

$$h_{ab} = g_{ab} + n_a n_b \quad (a, b, ... = 0, 1, 2, 3). \tag{3.3}$$

A timelike vector field t^a, which satisfies the condition $t^a \nabla_a t = 1$, will be introduced. This vector field may be decomposed into a lapse and shift, respectively N and N^a, which are

normal and tangential components with respect to the hypersurface, i.e., $N = -t^a n_a$ and $N_a = h_{ab} t^b$. The spatial three-metric h_{ab} and its conjugate momentum Π^{ab} correspond to dynamical degrees of freedom, while the lapse N and shift N_a are gauge variables with respect to the Hamiltonian and momentum constraints. By construction, it is obvious that the time-time and space-time components of the spatial metric vanish, i.e., $h_{tt} = h_{it} = h^{tt} = h^{it} = 0 \quad (i, j, ... = 1, 2, 3)$. Therefore, the ADM decomposition of the space-time metric g_{ab} is

$$g_{tt} = N_i N^i - N^2, \quad g_{it} = N_i,$$

and

$$g_{ij} = h_{ij}, \tag{3.4}$$

and possesses contravariant components

$$g^{tt} = -\frac{1}{N^2}, \quad g^{it} = \frac{N^i}{N^2}$$

and

$$g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \tag{3.5}$$

Analogous to the Vlasov-Maxwell system, the time coordinate is a "natural" coordinate corresponding to a timelike Killing field. An equilibrium solution of the Vlasov-Einstein system implies that the distribution function, the metric, and the conjugate momentum tensor are all independent of time t. A Lie bracket $\langle F, G\rangle$, acting on functional pairs $F[h_{ab}, \Pi^{ab}, f]$ and $G[h_{ab}, \Pi^{ab}, f]$, will define the cosymplectic structure of the Hamiltonian formulation. For the fundamental dynamical variables h_{ab}, Π^{ab}, and f, we take $\langle F, G\rangle$ to possess the form

$$\langle F, G\rangle[h_{ab}, \Pi^{ab}, f] = 16\pi \int d^3x \left(\frac{\delta F}{\delta h_{ab}} \frac{\delta G}{\delta \Pi^{ab}} - \frac{\delta G}{\delta h_{ab}} \frac{\delta F}{\delta \Pi^{ab}} \right)$$

$$+\int d\Gamma f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}, \tag{3.6}$$

where

$$\{a,b\} = \frac{\partial a}{\partial x^i}\frac{\partial b}{\partial p_i} - \frac{\partial b}{\partial x^i}\frac{\partial a}{\partial p_i}$$

denotes the standard canonical Poisson bracket, and $\delta/\delta X$ denotes the functional derivative with respect to the variable X. For a t = constant hypersurface, the covariant phase space volume element can be written as

$$d\Gamma = d^3x d^3p dm. \tag{3.7}$$

If d^3p refers to spatial momentum, as defined in the cotangent bundle, then d^3xd^3p is covariant. Therefore, the spatial coordinates x^i, the spatial momentum p_i, and the mass m are held fixed with respect to variations carried out at some time t. Consequently, the volume elements $d\Gamma$ and d^3x are not subject to variation, i.e., $\delta d\Gamma = \delta d^3x = 0$. Kandrup and Morrison (1993) discuss other possible choices in greater detail (e.g., the original Ipser-Thorne (1968) prescription).

It is straightforward to show that the bracket is antisymmetric and satisfies the Jacobi identity

$$\langle F,\langle G,H\rangle\rangle + \langle G,\langle H,F\rangle\rangle + \langle H,\langle F,G\rangle\rangle = 0. \tag{3.8}$$

Therefore, it is a *bona fide* Lie bracket. Consequently, the dynamics associated with this bracket and any Hamiltonian H will be symplectic with regard to the infinite-dimensional phase space (h_{ab}, Π^{ab}, f). The bracket (3.6) is an obvious analog to the bracket for the curved space Vlasov-Maxwell system. In the absence of matter, the bracket $\langle F, G\rangle$ corresponds to the natural bracket of vacuum gravity (cf. appendix E in Wald (1984)). In the case of a spherically symmetric space-time, the metric can be written as a functional of the distribution f and eliminated as a dynamical variable. Consequently, the bracket $\langle F, G\rangle$

will reduce to the bracket $[F, G]$ used by Kandrup and Morrison (1993). It is relatively straightforward to identify the Hamiltonian function H. The Hamiltonian will be the sum of a purely gravitational piece H_G and a matter contribution H_M. The Hamiltonian of vacuum gravity (cf. Arnowitt, Deser, and Misner (1962)) has the form

$$H_G = \int d^3x \mathcal{H}_G,$$

where

$$\mathcal{H}_G = \frac{1}{16\pi} h^{1/2} \{ N[-^{(3)}R + h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2)] \\ -2N_b[D_a(h^{-1/2}\Pi^{ab})] + 2D_a(h^{-1/2}N_b\Pi^{ab})\}. \quad (3.9)$$

In terms of notation, the covariant derivative operator is denoted D_a and the scalar curvature of the spatial metric h_{ab} is denoted ${}^{(3)}R$. The matter Hamiltonian is taken to be the minimal coupling of distribution to particle energy, viz.

$$H_M = \int d^3x \mathcal{H}_M = \int d\Gamma f \varepsilon, \quad (3.10)$$

where the ordinary particle energy is denoted $\varepsilon = -t^a p_a$. For appropriate "time" coordinates, the particle energy is $\varepsilon = -p_t$. The fundamental momentum variables, which appear in equation (3.10), are the spatial momentum components p_i and the particle mass m. The particle energy ε is a function of the variables $\{x^i, p_i, m, t\}$ and may be defined by the mass shell condition,

$$-\frac{\varepsilon^2}{N^2} - \frac{2\varepsilon N^a p_a}{N^2} + \left(h^{ab} - \frac{N^a N^b}{N^2}\right) p_a p_b = -m^2. \quad (3.11)$$

An alternative form of H_M is

$$H_M = \int N h^{1/2} d^3x T_t{}^t,$$

where

$$T^t{}_t = \int \frac{d^4p}{(-g)^{1/2}} \frac{f}{m} p^t p_t = \int \frac{d^3p dp_t}{N h^{1/2}} \frac{f}{m} p^t p_t \quad (3.12)$$

is the t-t component of the stress-energy tensor written as a momentum integral of the distribution. By variation of the gauge variables N and N_a, one can demonstrate that the Hamiltonian and momentum constraints hold. Performing the calculation

$$\frac{\partial \varepsilon}{\partial N} = \frac{(\varepsilon + N^a p_a)^2}{-N^3 p^t}, \tag{3.13}$$

enables one to determine the functional derivative

$$\begin{aligned} \frac{\delta H_M}{\delta N} &= \int \frac{d^3p dm}{N} \left[\frac{(\varepsilon + N^a p_a)^2}{-N^2 p^t} \right] f \\ &= h^{1/2} \int \frac{d^3p dp_t}{N h^{1/2}} \left[\frac{(\varepsilon + N^a p_a)^2}{N^2} \right] \frac{f}{m} = h^{1/2} T_{ab} n^a n^b, \end{aligned} \tag{3.14}$$

where n^a is the $t =$ constant hypersurface unit normal vector. The identity of the form $n^a = (1/N)(t^a - N^a)$ has been used in the final equality. The condition $\delta H/\delta N = 0$, i.e., that H is extremal with respect to variations of the lapse N, must hold. This may be calculated by combining equation (3.12) with $\delta H_G/\delta N$. The final result is the Hamiltonian constraint equation

$$G_{ab} n^a n^b = 8\pi T_{ab} n^a n^b, \tag{3.15}$$

which may be rendered as

$$-{}^{(3)}R + h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) = 16\pi \int \frac{d^3p dp_t}{N h^{1/2}} \left[\frac{(\varepsilon + N^a p_a)^2}{N^2} \right] \frac{f}{m}. \tag{3.16}$$

Likewise, one may calculate the result

$$\frac{\delta H_M}{\delta N_a} = -h^{1/2} \int \frac{d^3p dp_t}{N h^{1/2}} \left[\frac{h^{ba} p_b (\varepsilon + N^c p_c)}{N} \right] \frac{f}{m}, \tag{3.17}$$

which leads to the momentum constraint $\delta H/\delta N_a = 0$,

$$h^{ab} G_{bc} n^c = 8\pi h^{ab} T_{bc} n^c, \tag{3.18}$$

that may be written as

$$2D_b(h^{-1/2}\Pi^{ab}) = -16\pi \int \frac{d^3p dp_t}{N h^{1/2}} \left[\frac{h^{ba} p_b (\varepsilon + N^c p_c)}{N} \right] \frac{f}{m}. \tag{3.19}$$

The initial value problem demands the imposition of these constraints. However, this is auxilliary to the dynamical aspects of the Hamiltonian formulation. Assuming that the dynamical equations are generated by the Hamiltonian H and Lie bracket $\langle .,. \rangle$, it is guaranteed that the constraint conditions will be enforced and propagated to later times. If the bracket and Hamiltonian correspond to a suitable Hamiltonian formulation, then the constraint equation

$$\partial_t F = \langle F, H \rangle, \tag{3.20}$$

which holds for arbitrary functionals F, must generate the dynamical equations of the Vlasov-Einstein system. The dynamical equations of the gravitational field can be investigated first. From computation these take the form

$$\partial_t h_{ab} = \langle h_{ab}, H \rangle = 16\pi \frac{\delta H}{\delta \Pi^{ab}} \tag{3.21a}$$

and

$$\partial_t \Pi^{ab} = \langle \Pi^{ab}, H \rangle = -16\pi \frac{\delta H}{\delta h_{ab}}. \tag{3.21b}$$

Along with the functional derivative

$$\frac{\delta H_M}{\delta h_{ab}} =$$

$$-\frac{1}{2} N h^{1/2} \int \frac{d^3p dp_t}{N h^{1/2}} \left[\left(h^{ac} h^{bd} - \frac{N^b N^c h^{ad}}{n^2} - \frac{N^b N^d h^{ac}}{n^2} \right) p_c p_d - \frac{2\varepsilon N^a h^{bd} p_d}{N^2} \right] \frac{f}{m}, \tag{3.22}$$

the momentum constraint (3.19) and the standard equations for $\delta H_G / \delta h_{ab}$ and $\delta H_G / \delta \Pi^{ab}$ (cf. Arnowitt, Deser and Misner (1962)) can be combined to yield the dynamical Einstein equations, viz.

$$\partial_t h_{ab} = 16\pi \frac{\delta H}{\delta \Pi^{ab}} = 2h^{-1/2} N (\Pi_{ab} - \frac{1}{2} h_{ab} \Pi) + 2 D_{(a} N_{b)}, \tag{3.23a}$$

and

$$\partial_t \Pi^{ab} = -16\pi \frac{\delta H}{\delta h_{ab}}$$

$$= -Nh^{1/2}({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}Rh^{ab}) + \frac{1}{2}Nh^{-1/2}h^{ab}(\Pi_{cd}\Pi^{cd} - \frac{1}{2}\Pi^2)$$

$$-2Nh^{-1/2}(\Pi^{ac}\Pi_c{}^b - \frac{1}{2}\Pi\Pi^{ab}) + h^{1/2}(D^aD^bN - h^{ab}D^cD_cN)$$

$$+h^{1/2}D_c(h^{-1/2}N^c\Pi^{ab}) - 2\Pi^{c(a}D_cN^{b)}$$

$$+8\pi Nh^{1/2}\int \frac{d^3p dp_t}{Nh^{1/2}}\frac{f}{m}h^{ac}p_cp_dh^{bd}. \tag{3.23b}$$

The Vlasov equation possesses the form

$$\partial_t f = (f, H\rangle = \{\varepsilon, f\}. \tag{3.23c}$$

This is just the 3+1 decomposition of the covariant Vlasov equation (3.1).

3.2 All Vlasov-Einstein Equilibria are Energy Extremals

A solution to the dynamical equations, $\{h^o{}_{ab}(x^i), \Pi_o{}^{ab}(x^i), f_o(x^i, p_i, m)\}$, that is time-independent, will be an equilibrium solution of the Vlasov-Einstein system. Consequently, one may conclude that $\partial_t h^o{}_{ab} = \partial_t\Pi_o{}^{ab} = \partial_t f_o = 0$. The aforementioned dynamical equations, combined with the conditions of equilibrium, imply that $\delta H/\delta h_{ab} = \delta H/\delta\Pi^{ab} = \{\varepsilon_o, f_o\} = 0$. The particle energy at equilibrium is denoted ε_o. Generic time-independent equilibria, $\{h^o{}_{ab}, \Pi_o{}^{ab}, f_o\}$, only constitute energy extremals with regard to the restricted class of dynamically accessible perturbations. However, one cannot assume that the first variation $\delta^{(1)}H$ vanishes for perturbations of an arbitrary type. The first variation $\delta^{(1)}H$ will take the general form

$$\delta^{(1)}H =$$

$$\int d^3x\Big(\frac{\delta\mathcal{H}}{\delta h_{ab}}\delta^{(1)}h_{ab} + \frac{\delta\mathcal{H}}{\delta\Pi^{ab}}\delta^{(1)}\Pi^{ab} + \frac{\delta\mathcal{H}}{\delta N}\delta^{(1)}N + \frac{\delta\mathcal{H}}{\delta N_a}\delta^{(1)}N_a + \frac{\delta\mathcal{H}}{\delta f}\delta^{(1)}f\Big). \tag{3.24}$$

Employing the dynamical equations and the conditions of constraint leads to

$$\delta^{(1)}H = \frac{1}{16\pi}\int d^3x(-\partial_t\Pi^{ab}\delta^{(1)}h_{ab} + \partial_t h_{ab}\delta^{(1)}\Pi^{ab}) + \int d\Gamma\varepsilon_o\delta^{(1)}f. \tag{3.25}$$

The equilibrium condition guarantees that $\partial_t \Pi_o{}^{ab} = \partial_t h^o{}_{ab} = 0$. The last term involving $\delta^{(1)} f$ will only vanish for a phase-preserving perturbation, i.e.,

$$\delta^{(1)} H = \int d\Gamma \varepsilon_o \{g, f_o\} = -\int d\Gamma g\{\varepsilon_o, f_o\} = 0, \tag{3.26}$$

where an integration by parts has been performed. The final equality follows from the equilibrium condition $\{\varepsilon_o, f_o\} = \partial_t f_o = 0$. Also, one should note that the condition $\delta^{(1)} H = 0$ can be used to define an equilibrium.

3.3 The Second Variation $\delta^{(2)} H$

The set of variables $\{h_{ab}, \Pi^{ab}, N, N_a, f\}$ can be collectively denoted as Y^I. The second variation $\delta^{(2)} H$ can be written in terms of this "book-keeping" notation as

$$\delta^{(2)} H = \sum_I \frac{\delta H}{\delta Y^I} \delta^{(2)} Y^I + \frac{1}{2} \sum_I \sum_J \frac{\delta^{(2)} H}{\delta Y^I \delta Y^J} \delta^{(1)} Y^I \delta^{(1)} Y^J, \tag{3.27}$$

where the functional derivatives are computed with respect to the equilibrium distribution. The second variation $\delta^{(2)} H$ provides a nontrivial criterion for linear stability. Nonlinear stability criteria can be obtained by analyzing arbitrarily high orders of variation.

Employing the conditions of constraint, $\partial\mathcal{H}/\partial N = \partial\mathcal{H}/\partial N_a = 0$, and the dynamical equations evaluated at a time-independent equilibrium, $\partial\mathcal{H}/\partial h_{ab} = \partial\mathcal{H}/\partial \Pi^{ab} = 0$, causes the first term of (3.27) to reduce to

$$\int d\Gamma \varepsilon_o \delta^{(2)} f. \tag{3.28}$$

Evaluating this term for a phase-preserving perturbation $\delta^{(2)} f$ of the form (2.18) yields the result

$$\int d\Gamma \varepsilon_o \delta^{(2)} f = \frac{1}{2} \int d\Gamma \varepsilon_o \{g, \{g, f_o\}\} = -\frac{1}{2} \int d\Gamma \{g, \varepsilon_o\}\{g, f_o\}. \tag{3.29}$$

Therefore, the second variation can be written as

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma\{g,\varepsilon_o\}\{g,f_o\} + \frac{1}{2}\sum_I\sum_J\int d\Gamma\frac{\partial^2(f\varepsilon)}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J$$
$$+\frac{1}{2}\sum_I\sum_J\frac{\delta^{(2)}H_G}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J, \tag{3.30}$$

where the vacuum Hamiltonian is denoted H_G. The second term in equation (3.30) can be rewritten as the following expression

$$\frac{1}{2}\sum_I\sum_J\int d\Gamma f\frac{\partial^2\varepsilon}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J + \sum_I\int d\Gamma\frac{\partial\varepsilon}{\partial Y^I}\delta^{(1)}Y^I\delta^{(1)}f, \tag{3.31}$$

which undergoes further reduction to the following format:

$$\frac{1}{2}\sum_I\int d\Gamma f\delta^{(1)}Y^I\delta^{(1)}\Big(\frac{\partial\varepsilon}{\partial Y^I}\Big) + \sum_I\int d\Gamma\delta^{(1)}f\delta^{(1)}Y^I\frac{\partial\varepsilon}{\partial Y^I}$$
$$=\sum_I\int d\Gamma\delta^{(1)}Y^I\Big[f\delta^{(1)}\Big(\frac{\partial\varepsilon}{\partial Y^I}\Big) + \delta^{(1)}f\frac{\partial\varepsilon}{\partial Y^I}\Big]$$
$$-\frac{1}{2}\sum_I\sum_J\int d\Gamma f\frac{\partial^2\varepsilon}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J = \sum_I\int d\Gamma\delta^{(1)}Y^I\delta^{(1)}\Big(f\frac{\partial\varepsilon}{\partial Y^I}\Big)$$
$$-\frac{1}{2}\sum_I\sum_J\int d\Gamma f\frac{\partial^2\varepsilon}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J. \tag{3.32}$$

The sum over variables Y^I in equation (3.32) does not include f or Π^{ab} because the particle energy ε does not depend on these variables. The first term of equation (3.32) can be written as

$$\int d^3x\Big[\delta\rho\delta N + \delta J^a\delta N_a + \frac{1}{2}\delta(NS^{ab} - 2N^aJ^b)\delta h_{ab}\Big], \tag{3.33}$$

where equation (3.13) for $\partial\varepsilon/\partial N$, and analogous expressions for $\partial\varepsilon/\partial N_a$ and $\partial\varepsilon/\partial h_{ab}$, have been used in the derivation. Equation (3.33) employs the notation

$$\rho = h^{1/2}n^aT_{ab}n^b, \quad J^a = h^{1/2}h^{ab}T_{bc}n^c,$$

and

$$S^{ab} = h^{1/2}h^{ac}T_{cd}h^{bd}. \tag{3.34}$$

First order variations $\delta\rho$, δJ^a and δS^{ab} are expressed in terms of $\delta^{(1)}f = \{g, f_o\}$. The second term in equation (3.32) may be evaluated by noting that

$$\frac{\partial^2\varepsilon}{\partial N^2} = \frac{\partial^2\varepsilon}{\partial N \partial N_a} = \frac{\partial^2\varepsilon}{\partial N_a \partial N_b} = 0, \tag{3.35a}$$

and that

$$\frac{\partial^2\varepsilon}{\partial h_{ab}\partial N} = \frac{-1}{2N(\varepsilon + N^a p_a)} N^2 h^{ac} h^{bd} p_c p_d, \tag{3.35b}$$

$$\frac{\partial^2\varepsilon}{\partial h_{ab}\partial N_c} = h^{c(a}h^{b)d} p_d, \tag{3.35c}$$

and

$$\frac{\partial^2\varepsilon}{\partial h_{ab}\partial h_{cd}} = -\frac{1}{4}\frac{N^4}{(\varepsilon + N^k p_k)^3} h^{ae}h^{bf}h^{cg}h^{dh} p_e p_f p_g p_h$$

$$+ \frac{N^2}{(\varepsilon + N^k p_k)} h^{f(b}h^{a)(c}h^{d)e} p_e p_f - 2h^{f(b}h^{a)(c}N^{d)} p_f. \tag{3.35d}$$

Use of equations (3.35b)-(3.35d) yields the expressions

$$-\int d\Gamma f \frac{\partial^2\varepsilon}{\partial h_{ab}\partial N} \delta h_{ab}\delta N = -\frac{1}{2}\int d^3x S^{ab}\delta h_{ab}\delta N, \tag{3.36}$$

$$-\int d\Gamma f \frac{\partial^2\varepsilon}{\partial h_{ab}\partial N_c} \delta h_{ab}\delta N_c = \int d^3x J^b h^{ac}\delta h_{ab}\delta N_c \tag{3.37}$$

and

$$-\int d\Gamma f \frac{\partial^2\varepsilon}{\partial h_{ab}\partial h_{cd}} \delta h_{ab}\delta h_{cd}$$

$$= \int d^3x \Big(\frac{1}{4} N S^{abcd} - N h^{ac} S^{bd} - 2h^{ac} J^b N^d\Big) \delta h_{ab}\delta h_{cd}, \tag{3.38a}$$

where

$$S^{abcd} = h^{1/2} h^{ae}h^{bf}h^{cg}h^{dh} \int \frac{d^4p}{(-g)^{1/2}} \frac{f}{m} \frac{p_e p_f p_g p_h}{(n^k p_k)^2}. \tag{3.38b}$$

Equations (3.36)-(3.38), together with equation (3.33), lead to the following expression for equation (3.32)

$$\int d^3x[\delta\rho\delta N + \delta J^a \delta N_a + \frac{1}{2}\delta(N S^{ab} - 2N^a J^b)\delta h_{ab} - \frac{1}{2} S^{ab}\delta h_{ab}\delta N$$

$$+J^b h^{ac}\delta h_{ab}\delta N_c + \left(\frac{1}{8}NS^{abcd} - \frac{1}{2}Nh^{ac}S^{bd} - h^{ac}J^bN^d\right)\delta h_{ab}\delta h_{cd}]. \tag{3.39}$$

One may use the identities

$$\delta(N^a J^b) = J^b h^{ac}\delta N_c + N^a \delta J^b - h^{ac}J^bN^d\delta h_{cd} \tag{3.40a}$$

and

$$\delta(NS^{ab}) = S^{ab}\delta N + N\delta S^{ab} \tag{3.40b}$$

to rewrite equation (3.39) in a more straightforward manner.

The use of equation (3.30) leads to the final result

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma\{g,\varepsilon_o\}\{g,f_o\}$$

$$+\int d^3x\left[\delta\rho\delta N + \delta J^a\delta N_a + \frac{1}{2}N\delta S^{ab}\delta h_{ab} - N^a\delta J^b\delta h_{ab}\right.$$

$$\left.+\left(\frac{1}{8}NS^{abcd} - \frac{1}{2}Nh^{ac}S^{bd}\right)\delta h_{ab}\delta h_{cd}\right] + \delta^{(2)}H_G[\delta^{(1)}Y], \tag{3.41a}$$

where

$$\delta^{(2)}H_G[\delta^{(1)}Y] = +\frac{1}{2}\sum_I\sum_J \frac{\delta^{(2)}H_G}{\delta Y^I\delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J, \tag{3.41b}$$

corresponds to the second variation $\delta^{(2)}H_G$ associated with the first order perturbation $\delta^{(1)}Y^I$. The matter contributions to $\delta^{(2)}H$ may be reformulated in terms of the distribution function. This alternate form can be written as

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma\{g,\varepsilon_o\}\{g,f_o\}$$

$$+\int d\Gamma\{g,f_o\}\left[\frac{1}{N}(\varepsilon + N^a p_a)\delta N - h^{ab}p_b\delta N_a\right.$$

$$\left.+\left(\frac{N^2}{2(\varepsilon + N^c p_c)}h^{ac}h^{bd}p_c p_d - N^a h^{bd}p_d\right)\delta h_{ab}\right]$$

$$+\int d\Gamma f\delta h_{ab}\delta h_{cd}\left[-\frac{N^4}{8(\varepsilon + N^k p_k)^3}h^{ae}h^{bf}h^{cg}h^{dh}p_e p_f p_g p_h\right.$$

$$-\frac{N^2}{2(\varepsilon+N^k p_k)}h^{ac}h^{de}h^{bf}p_e p_f + h^{ac}h^{bf}N^d p_f\Bigg]$$

$$+\delta^{(2)}H_G[\delta^{(1)}Y]. \tag{3.42}$$

A routine calculation of the gravitational contribution yields

$$16\pi\delta^{(2)}H_G[\delta^{(1)}Y] =$$

$$\int h^{1/2}d^3x\{[(^{(3)}R^{ab}-\frac{1}{2}h^{ab(3)}R)-\frac{1}{2}h^{-1}h^{ab}(\Pi_{cd}\Pi^{cd}-\frac{1}{2}\Pi^2)$$

$$+2h^{-1}(\Pi^{ac}\Pi_c{}^b-\frac{1}{2}\Pi\Pi^{ab})]\delta N\delta h_{ab}+2h^{-1}(\Pi_{ab}-\frac{1}{2}h_{ab}\Pi)\delta N\delta\Pi^{ab}$$

$$-(D^aD^b\delta N-h^{ab}D^cD_c\delta N)\delta h_{ab}\}$$

$$+\int d^3x\{h^{1/2}D_c[h^{-1/2}(2\Pi^{ac}\delta N^b-\Pi^{ab}\delta N^c)]\delta h_{ab}+2\delta\Pi^{ab}D_a\delta N_b\}$$

$$+\frac{1}{2}\int d^3x(ND^2H-N_aD^2\Delta^a), \tag{3.43}$$

where D^2H and $D^2\Delta^a$ correspond to the following expressions:

$$D^2H = h^{-1/2}\times$$

$$\{[\Pi^{ab}\Pi_{ab}-\frac{1}{2}\Pi^2][\frac{1}{4}(\delta h^c{}_c)^2+\frac{1}{2}\delta h^{cd}\delta h_{cd}]-2\Pi^{ab}\Pi^{cd}(h_{bd}\delta h_{ac}-\frac{1}{2}h_{cd}\delta h_{ab})\delta h^e{}_e$$

$$-2(\Pi_{ab}\delta\Pi^{ab}-\frac{1}{2}\Pi\delta\Pi)\delta h^c{}_c+2\Pi^{ab}\Pi^{cd}(\delta h_{ac}\delta h_{bd}-\frac{1}{2}\delta h_{ab}\delta h_{cd})$$

$$+2\delta\Pi^{ab}\delta\Pi^{cd}(h_{ac}h_{bd}-\frac{1}{2}h_{ab}h_{cd})$$

$$+2(h_{bd}\delta h_{ac}+h_{ac}\delta h_{bd}-\frac{1}{2}h_{cd}\delta h_{ab}-\frac{1}{2}h_{ab}\delta h_{cd})(\Pi^{cd}\delta\Pi^{ab}+\Pi^{ab}\delta\Pi^{cd})\}$$

$$-h^{1/2}\{^{(3)}R\Big[\frac{1}{4}(\delta h^a{}_a)^2-\frac{1}{2}\delta h^{ab}\delta h_{ab}\Big]-{}^{(3)}R_{ab}\delta h^{ab}\delta h^c{}_c$$

$$+2^{(3)}R_{ab}\delta h^{ac}\delta h_c{}^b-\delta h^{ab}(D^cD_b\delta h_{ac}+D^cD_a\delta h_{bc})+\delta h^c{}_c(D^bD^a\delta h_{ab}-D^bD_b\delta h^a{}_a)$$

$$+2\delta h^{cd}(D^aD_a\delta h_{cd}+D_dD_c\delta h^a{}_a-D_dD^a\delta h_{ac})$$

$$+\Big[\frac{1}{2}D^c\delta h^b{}_b-D_b\delta h^{bc}\Big](2D^a\delta h_{ac}-D_c\delta h^a{}_a)$$

$$+\frac{1}{2}D_a\delta h_{cd}D^a\delta h^{cd} + D^d\delta h_a{}^c(D_d\delta h_c{}^a - D_c\delta h_d{}^a)\} \tag{3.44a}$$

and

$$D^2\Delta^a = 2(\delta\Pi^{bc}h^{ad} - \Pi^{bc}\delta h^{ad})(D_c\delta h_{bd} + D_b\delta h_{cd} - D_d\delta h_{bc}). \tag{3.44b}$$

These formulae are given in the appendix of Moncrief (1976). The perturbed constraint equations permit a further simplification of the second variation $\delta^{(2)}H_G$.

The exact Hamiltonian constraint can be written

$$^{(3)}R + h^{-1}(\frac{1}{2}\Pi^2 - \Pi_{ab}\Pi^{ab}) = 16\pi h^{-1/2}\rho, \tag{3.45}$$

where ρ is the energy density. A linearized perturbation of this constraint will have the form

$$\begin{aligned}
&-\delta h_{ab}{}^{(3)}R^{ab} - D_cD^c\delta h_d{}^d + D^aD^b\delta h_{ab} \\
&+h^{-1}h^{cd}\delta h_{cd}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) - 2h^{-1}\delta\Pi^{ab}(\Pi_{ab} - \frac{1}{2}h_{ab}\Pi) \\
&-2h^{-1}\delta h_{ab}(\Pi^b{}_c\Pi^{ac} - \frac{1}{2}\Pi\Pi^{ab}) = 16\pi h^{-1/2}\delta\rho - 8\pi h^{-1/2}\rho h^{ab}\delta h_{ab}.
\end{aligned} \tag{3.46}$$

The constraint (3.45) may be used to eliminate the unperturbed ρ from this equation, giving the expression

$$\begin{aligned}
&(-^{(3)}R^{ab} + \frac{1}{2}h^{ab(3)}R)\delta h_{ab} - h^{ab}D_cD^c\delta h_{ab} + D^aD^b\delta h_{ab} \\
&+\frac{1}{2}h^{-1}h^{cd}\delta h_{cd}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) - 2h^{-1}\delta\Pi^{ab}(\Pi_{ab} - \frac{1}{2}h_{ab}\Pi) \\
&-2h^{-1}\delta h_{ab}(\Pi^b{}_c\Pi^{ac} - \frac{1}{2}\Pi\Pi^{ab}) = 16\pi h^{-1/2}\delta\rho.
\end{aligned} \tag{3.47}$$

Likewise, for the exact momentum constraint

$$D_a(h^{-1/2}\Pi^{ab}) = 8\pi h^{-1/2}J^b, \tag{3.48}$$

one may write its linearised perturbation as

$$2D_a\delta\Pi^{ab} + \Pi^{ac}h^{bd}(2D_c\delta h_{da} - D_d\delta h_{ac}) = 16\pi\delta J^b. \tag{3.49}$$

The perturbed constraint equations (3.47) and (3.49) may be used to reduce equation (3.43) to the form

$$\delta^{(2)}H_G[\delta^{(1)}Y^I] = -\int d^3x(\delta\rho\delta N + \delta J^a\delta N_a)$$
$$+\frac{1}{32\pi}\int d^3x(ND^2H - N_aD^2\Delta^a). \tag{3.50}$$

This expression may be substituted into equation (3.41a) to obtain the final form for $\delta^{(2)}H$, viz.

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma\{g,\varepsilon_o\}\{g,f_o\}$$
$$+\int d^3x\Big[\frac{1}{2}N\delta S^{ab}\delta h_{ab} - N^a\delta J^b\delta h_{ab}$$
$$+\Big(\frac{1}{8}NS^{abcd} - \frac{1}{2}Nh^{ac}S^{bd}\Big)\delta h_{ab}\delta h_{cd}\Big]$$
$$+\frac{1}{32\pi}\int d^3x(ND^2H - N_aD^2\Delta^a). \tag{3.51}$$

This formalism can be used to study the stability of spherical equilibria with respect to non-radial perturbations. With the notable exception of Ipser and Semenzato (1979), little work has been devoted to this problem. Consequently, the subject is not well understood. However, the energy functional (3.41b) should be, in principle, amenable to the testing of non-radial perturbations that cause $\delta^{(2)}H < 0$. Also, the formalism can be used to test more general spherical equilibria than in Ipser and Semenzato (1979). These authors, in their development of the symmetric operator formalism mentioned in the introduction, restricted the equilibrium to the form $f_o = f_o(\varepsilon, m)$, where ε and m correspond, respectively, to the particle energy and mass. A generic spherical equilibrium may be written as $f_o = f_o(\varepsilon, J^2, m)$, where the squared angular momentum, associated with rotational symmetry, is denoted J^2. Ipser and Semenzato (1979) also imposed the restriction that the equilibrium has a monotonically decreasing dependence on ε. Therefore, the derivative $\partial f_o/\partial\varepsilon$ is intrinsically negative with respect to all values of the particle energy ε. The

formalism developed here does not place constraints on the form of the equilibrium distribution. Another objective is the study of the stability of axisymmetric equilibria. Little research has been carried out on this subject. The motivation for this work comes from the fact that physical equilibria generally possess angular momentum. Consequently, there should be a flattening of the matter distribution due to the effects of rotation. Shapiro and Teukolsky (1993a,b) have succeeded in generating axisymmetric equilibria through a useful numerical algorithm which they have recently developed.

It is plausible to conjecture that the stability of rotating, axisymmetric systems can be studied with the formalism developed here. One may attempt to prove that certain classes of equilibria always possess phase-preserving perturbations for which the energy decreases. Subsequently, further investigation may explain if, for relatively short time scales, these perturbations result in physical instabilities.

Unfortunately, it is not straightforward to follow through with this program. Attacking the problem by means of a post-Newtonian expansion leads to inconclusive results. It appears that a generic rotating, axisymmetric solution of the Einstein equations must be found before stability results can be deduced from it. However, such a solution is not known.

CHAPTER 4
THE VLASOV-BRANS-DICKE SYSTEM

Carl Brans and Robert Dicke (1961), in an attempt to make Mach's principle compatible with General Relativity, formulated a scalar-tensor theory of gravity in which the Newtonian gravitational constant G_o is represented by a scalar field. This is the celebrated Brans-Dicke gravitational theory. It was shown by Dicke (1962) that Brans-Dicke theory may be written in an equivalent representation in which the inertial masses of elementary particles vary as a function of the Brans-Dicke scalar field while the gravitational constant G_o remains fixed. The Brans-Dicke theory may be formulated in either representation. Although the physical interpretation of gravitational phenomenon varies between the two representations, the actual physical predictions remain equivalent.

For example, as pointed out by Dicke (1962), the gravitational red-shift in the variable mass representation is only a partially metric phenomenon. Since the particle masses are affected by the scalar field, part of the red-shift results from changes in the energy levels of the atoms. Consequently, the red-shift is not entirely metric induced. In this representation, the measures provided by rulers and clocks are not invariant with respect to position in space-time. Also, free falling matter does not follow geodesic trajectories in space-time. However, (massless) photons still do. The geodesic equation of the variable G representation is

$$\frac{d}{d\tau}(mg_{ij}u^j) - \frac{1}{2}mg_{jk,i}u^j u^k = 0, \tag{4.1}$$

where τ denotes the proper time and u^i is the relativistic four-velocity. One can switch to

the variable mass representation through the transformation

$$\bar{m} = \phi^{1/2} m \tag{4.2}$$

and

$$\bar{g}_{ij} = \phi^{-1} g_{ij}. \tag{4.3}$$

The speed of light c (units: $[L][T]^{-1}$) and Planck's constant $\hbar$ (units: $[M][L]^2\,[T]^{-1}$) remain invariant under this transformation. Consequently, the geodesic equation for the variable mass representation has the form

$$\frac{d}{d\tau}(\bar{m}\bar{g}_{ij}\bar{u}^j) - \frac{1}{2}\bar{m}\bar{g}_{jk,i}\bar{u}^j\bar{u}^k + \bar{m}_{,i} = 0. \tag{4.4}$$

As shown by Toton (1970), the original Brans-Dicke variational principle

$$\delta \int d^4x(-g)^{1/2}(\phi R - \omega\frac{\phi_{,\mu}\phi^{,\mu}}{\phi} + 16\pi\frac{G_o}{c^4}L_M) = 0 \tag{4.5}$$

may be reformulated, by means of the conformal map (4.3), as

$$\delta \int d^4x(-g)^{1/2}(R - \left(\frac{3}{2} + \omega\right)\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2} + 16\pi\frac{G_o}{c^4}\frac{1}{\phi^2}\bar{L}_M) = 0, \tag{4.6}$$

where $\bar{L}_M$ is obtained from L_M by replacing $g_{\mu\nu}$ in L_M by $g_{\mu\nu}/\phi$. The field equations associated with the original Brans-Dicke variational principle (4.5) are

$$\phi G_{\mu\nu} = \frac{8\pi G_o}{c^4}T_{\mu\nu} + \phi_{;\mu\nu} - g_{\mu\nu}\phi^{;\alpha}{}_{;\alpha} + \frac{\omega}{\phi}[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha}] \tag{4.7}$$

and

$$(2\omega + 3)\phi^{;\alpha}{}_{;\alpha} = \frac{8\pi G_o}{c^4}T^{\alpha}{}_{\alpha}, \tag{4.8}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. As noted by Bruckman and Velázquez (1993), the presence of second order derivatives of ϕ in (4.7) make the variable G representation of Brans-Dicke theory unsuitable for application of the canonical ADM Hamiltonian formalism.

Through examination of the field equations associated with the variational principle (4.6), viz.

$$G_{\mu\nu} = \frac{8\pi G_o}{c^4}\frac{\bar{T}_{\mu\nu}}{\phi} + \left(\frac{3}{2}+\omega\right)(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi^{,\alpha}\Phi_{,\alpha}) \tag{4.9}$$

and

$$\left(\frac{3}{2}+\omega\right)\nabla^{\mu}\nabla_{\mu}\phi - \left(\frac{3}{2}+\omega\right)\frac{\nabla^{\mu}\phi\nabla_{\mu}\phi}{\phi} - \frac{16\pi G_o}{c^4}\frac{1}{\phi}\bar{L}_M + \frac{8\pi G_o}{c^4}\frac{\delta\bar{L}_M}{\delta\phi} = 0, \tag{4.10}$$

where

$$\Phi = \ln\phi, \tag{4.11}$$

one may see that equation (4.9) has the form of the standard Einstein field equation. Equation (4.9) can be written as

$$G_{\mu\nu} \equiv \frac{8\pi G_o}{c^4}\bar{T}^{tot}_{\mu\nu}, \tag{4.12}$$

where $\bar{T}^{tot}_{\mu\nu}$ is the sum of a matter stress-energy tensor and a scalar field pseudo stress-energy tensor. Consequently, because of the similarity to Einstein theory, the ADM formalism is easily applied to the variable mass representation of the theory.

Following Kandrup and O'Neill (1994), we may, while employing an ADM splitting into space and time, demonstrate that the collisionless Boltzmann, i.e., Liouville equation of the Brans-Dicke gravitational theory is Hamiltonian. By analogy with the Vlasov-Einstein case, criteria for the linear stability of time-independent equilibria, corresponding to relativistic matter configurations, will be derived.

As for notation, all physical quantities subjected to the transformations (4.2) and (4.3) will be denoted with a bar superscript. This notation is employed in the equations (4.2) and (4.3). Unbarred quantities, of course, have not been subjected to this transformation. In terms of physical variables, one works with a spatial three-metric h_{ab} and scalar field ϕ, as

well as their respective conjugate momenta Π^{ab} and Π. These variables are sufficient for the vacuum metric-scalar Hamiltonian formalism. In the presence of matter, one must include the distribution function f as a fifth variable. Actually, as will be shown, it is necessary to work with the barred distribution $\bar{f}$. Consequently, the Hamiltonian formalism is written in terms of a mixed system of variables. One may define functionals $F = F[h_{ab}, \Pi^{ab}; \phi, \Pi; \bar{f}]$ in terms of an infinite-dimensional phase space $(h_{ab}, \Pi^{ab}; \phi, \Pi; \bar{f})$.

It will be shown that the Vlasov-Brans-Dicke dynamical equations are equivalent to the constraint equations $\partial_t F = \langle F, H \rangle$, where F denotes a phase space functional, ∂_t denotes a coordinate time derivative, H denotes a Hamiltonian and $\langle ., . \rangle$ denotes a bracket operation, to be later defined, which acts on the functional pairs $F[h_{ab}, \Pi^{ab}; \phi, \Pi; \bar{f}]$ and $G[h_{ab}, \Pi^{ab}; \phi, \Pi; \bar{f}]$.

One may demonstrate that the first order variation is extremal,

$$\delta^{(1)} H[\delta^{(1)} X] = 0, \tag{4.13}$$

when subjected to a perturbation

$$\delta^{(1)} X = \{\delta^{(1)} h_{ab}, \delta^{(1)} \Pi^{ab}; \delta^{(1)} \phi, \delta^{(1)} \Pi; \delta^{(1)} \bar{f}\} \tag{4.14}$$

that is dynamically accessible. This perturbation will be consistent with both the Hamiltonian and momentum constraint equations and with matter field constraints due to conservation of phase. The expression $\delta^{(2)} H$ may be computed, the sign of which is related to the stability of the time-independent equilibrium. A positive sign, $\delta^{(2)} H > 0$, guarantees that the system is stable. A negative sign, $\delta^{(2)} H < 0$, does not guarantee linear instability, but should, at least, ensure nonlinear nonstability or nonstability with respect to dissipative effects.

4.1 The Vlasov-Brans-Dicke Hamiltonian Formulation

The ADM decomposition into space and time will be implemented in exactly the same way as in Kandrup and O'Neill (1994), which, in turn, followed the treatment given in Wald (1984). The metric line element will transform according to equation (4.2) as

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \phi^{-1} g_{\mu\nu} dx^\mu dx^\nu = \phi^{-1} ds^2. \tag{4.15}$$

Therefore, the transformed metric line element is

$$d\bar{s} = \phi^{-1/2} ds. \tag{4.16}$$

The expression (4.16) is the same as Dicke's (1962) $\lambda^{-1/2} d\bar{s} = ds$, where the arbitrary well-behaved function λ is set equal to ϕ^{-1}. Consequently, the relativistic four-velocity transforms as

$$\bar{u}^\mu = \phi^{1/2} u^\mu. \tag{4.17}$$

The expression (4.2) enables one to derive an expression for the transformed four-momentum, viz.

$$\bar{p}^\mu = \phi p^\mu. \tag{4.18}$$

The covariant form for the transformed four-momentum is given by

$$\bar{p}_\mu = p_\mu, \tag{4.19}$$

which implies that the covariant four-momentum is the same with respect to both representations of the Brans-Dicke theory. In the Vlasov-Einstein case (Kandrup and O'Neill (1994)), one worked with a distribution function of the form $f = f(x^\alpha, p_\alpha, m, t)$, where x^α denotes the spatial coordinates, p_α denotes the covariant spatial momenta, m denotes the particle mass and t is the time. For the variable mass representation of Brans-Dicke theory,

the distribution will be

$$\bar{f} = f(\bar{x}^\alpha, \bar{p}_\alpha, \bar{m}, \bar{t}) = f(x^\alpha, p_\alpha, \bar{m}, t). \tag{4.20}$$

It should be noted that only the mass variable m is altered by the transformation equations (4.2) and (4.3). The scalar field ϕ cannot be factored out of the distribution (4.20), implying that the barred distribution $\bar{f}$ must be used in the Hamiltonian formulation.

One can postulate the following bracket

$$\begin{aligned}\langle F, G\rangle[h_{ab}, \Pi^{ab}; \phi, \Pi; \bar{f}] = 16\pi \int d^3x \left(\frac{\delta F}{\delta h_{ab}}\frac{\delta G}{\delta \Pi^{ab}} - \frac{\delta F}{\delta \Pi_{ab}}\frac{\delta G}{\delta h^{ab}}\right) \\ +16\pi \int d^3x \left(\frac{\delta F}{\delta \phi}\frac{\delta G}{\delta \Pi} - \frac{\delta F}{\delta \Pi}\frac{\delta G}{\delta \phi}\right) \\ +16\pi \int d\bar{\Gamma} \bar{f} \left\{\frac{\delta F}{\delta \bar{f}}, \frac{\delta G}{\delta \bar{f}}\right\}_{\bar{x},\bar{p}},\end{aligned} \tag{4.21}$$

where $\delta/\delta X$ denotes the usual functional derivative with respect to the variable X, and

$$\{A, B\}_{\bar{x},\bar{p}} = \frac{\partial A}{\partial \bar{x}^a}\frac{\partial B}{\partial \bar{p}_a} - \frac{\partial B}{\partial \bar{x}^a}\frac{\partial A}{\partial \bar{p}_a} \tag{4.22}$$

is the ordinary canonical Poisson bracket written in terms of the variable mass representation. Other notations in equations (4.21) and (4.22) are as in Kandrup and O'Neill (1994). It is interesting to note that the bracket is of a mixed composition, with the matter component written in terms of the transformed variables.

The bracket (4.21) is a *bona fide* Lie bracket because (a) it is antisymmetric and (b) it satisfies the Jacobi identity

$$\langle F, \langle G, H\rangle\rangle + \langle G, \langle H, F\rangle\rangle + \langle H, \langle F, G\rangle\rangle = 0. \tag{4.23}$$

We may separate the Hamiltonian function into two parts: one part, H_G, which is purely gravitational and another part, H_M, which governs the matter coupling to the gravitational field. Variation of the action given by Brans and Dicke (1961), leads to the Brans-Dicke

field equations written in the variable G representation. Subjecting these equations to the transformations (4.2) and (4.3) yields an altered form of them. Due to the fact that the scalar field is determined by the field equations, the quantitative physical predictions of the theory remain the same. The variable mass representation of the Brans-Dicke action (4.6) does not couple the Ricci scalar invariant R to the scalar field ϕ. Therefore, when the ADM formalism is implemented, one can use partial integration to rid the action of unwanted boundary terms in a straightforward manner. Thus, it is convenient to work within the variable mass representation.

Following the treatment given by Toton (1970), one can write H_G as

$$H_G = \int d^3x \mathcal{H}_G, \tag{4.24}$$

where the Hamiltonian density is

$$\begin{aligned} \mathcal{H}_G = \frac{c^4}{16\pi G_o} h^{1/2}\{N[-{}^{(3)}R + h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) + \frac{h^{-1}\phi^2\Pi^2}{(6+4\omega)} \\ + \left(\frac{3}{2}+\omega\right)\frac{h^{ab}\phi_{,a}\phi_{,b}}{\phi^2}] - 2N_b[D_a(h^{-1/2}\Pi^{ab}) - \frac{1}{2}h^{-1/2}\Pi\phi^{,b}]\}, \end{aligned} \tag{4.25}$$

where D_a denotes the covariant derivative associated with the spatial three-metric h_{ab} and ${}^{(3)}R$ denotes the three-curvature scalar corresponding to h_{ab}. The matter Hamiltonian, written with respect to the variable mass representation, takes the form

$$\bar{H}_M = \int d\bar{\Gamma} \bar{f} \bar{\varepsilon}. \tag{4.26}$$

If it is demanded that the Hamiltonian H be extremal with respect to variations in the lapse N and shift N_a, then the Hamiltonian and momentum constraints are enforced. Following the treatment given in Kandrup and O'Neill (1994), one may derive the expression

$$\frac{\partial \varepsilon}{\partial N} = \frac{(\varepsilon + N^a p_a)^2}{-N^3 p^t}, \tag{4.27}$$

which implies that

$$\frac{\delta H_M}{\delta N} = \int \frac{d^3\bar{p}d\bar{m}}{N}\left[\frac{(\varepsilon + N^a p_a)^2}{-N^2 p^t}\right]\bar{f} = \frac{\bar{h}^{1/2}}{\phi^{1/2}}\int \frac{d^3\bar{p}d\bar{p}_t}{\bar{N}\bar{h}^{1/2}}\left[\frac{(\bar{\varepsilon} + \bar{N}^a\bar{p}_a)^2}{\bar{N}^2}\right]\frac{\bar{f}}{\bar{m}}$$
$$= \frac{\bar{h}^{1/2}}{\phi^{1/2}}\bar{T}_{ab}\bar{n}^a\bar{n}^b, \tag{4.28}$$

where $\bar{n}^a$ is a unit normal vector orthogonal to a t = constant hypersurface. The final equality follows from the identity $\bar{n}^a = (\bar{t}^a - \bar{N}^a)/\bar{N}$. It should be noted that the derivation of equation (4.28) has used the relation $\bar{\varepsilon} = \varepsilon$. Likewise, one may derive an expression for $\delta H_G/\delta N$, viz.

$$\frac{\delta H_G}{\delta N} = \frac{c^4}{16\pi G_o}h^{1/2}\left[-{}^{(3)}R + h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) + \frac{h^{-1}\phi^2\Pi^2}{(6+4\omega)} + \left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2}\right]. \tag{4.29}$$

The Hamiltonian constraint, $\delta H/\delta N = 0$, is obtained by combining equations (4.28) and (4.29), which yields the expression

$$h^{1/2}\left[-{}^{(3)}R + h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) + \frac{h^{-1}\phi^2\Pi^2}{(6+4\omega)} + \left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2}\right]$$
$$= \frac{16\pi G_o}{c^4}\frac{\bar{h}^{1/2}}{\phi^{1/2}}\bar{T}_{ab}\bar{n}^a\bar{n}^b. \tag{4.30}$$

One may check the Hamiltonian constraint (4.30) by deriving it from the field equation of the variable mass representation (4.9)

$$G_{\mu\nu} = \frac{8\pi G_o}{c^4}\frac{\bar{T}_{\mu\nu}}{\phi} + \left(\frac{3}{2}+\omega\right)\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi^{,\alpha}\Phi_{,\alpha}\right). \tag{4.31}$$

Taking the double inner product with the unit normal vector n^μ gives

$$G_{\mu\nu}n^\mu n^\nu + \left(\frac{3}{2}+\omega\right)\left(\frac{1}{2}g_{\mu\nu}n^\mu n^\nu\Phi^{,\alpha}\Phi_{,\alpha} - \Phi_{,\mu}n^\mu\Phi_{,\nu}n^\nu\right)$$
$$-\frac{8\pi G_o}{c^4}\frac{\bar{T}_{\mu\nu}}{\phi}n^\mu n^\nu = 0. \tag{4.32}$$

Since $\bar{n}^\mu$ transforms as

$$\bar{n}^\mu = n^\mu\sqrt{\phi} = \frac{1}{\bar{N}}(\bar{t}^\mu - \bar{N}^\mu), \tag{4.33}$$

where

$$\bar{t}^{\mu} = t^{\mu} \tag{4.34a}$$

$$\bar{N}^{\mu} = N^{\mu} \tag{4.34b}$$

$$\frac{1}{\bar{N}} = \frac{\sqrt{\phi}}{N}, \tag{4.34c}$$

one may rewrite equation (4.32) as

$$G_{\mu\nu}n^{\mu}n^{\nu} + \left(\frac{3}{2} + \omega\right)\left(\frac{1}{2}g_{\mu\nu}n^{\mu}n^{\nu}\Phi^{,\alpha}\Phi_{,\alpha} - \Phi_{,\mu}n^{\mu}\Phi_{,\nu}n^{\nu}\right)$$
$$-\frac{8\pi G_o}{c^4}\frac{\bar{T}_{\mu\nu}}{\phi^2}\bar{n}^{\mu}\bar{n}^{\nu} = 0. \tag{4.35}$$

Substitution of the following relation

$$G_{\mu\nu}n^{\mu}n^{\nu} = \frac{1}{2}[{}^{(3)}R + h^{-1}(\frac{\Pi^2}{2} - \Pi^{ab}\Pi_{ab})] \tag{4.36}$$

into equation (4.35) yields the resultant expression

$$\frac{1}{2}[{}^{(3)}R + h^{-1}(\frac{\Pi^2}{2} - \Pi^{ab}\Pi_{ab})] + \left(\frac{3}{2} + \omega\right)\left(\frac{1}{2}g_{\mu\nu}n^{\mu}n^{\nu}\Phi^{,\alpha}\Phi_{,\alpha} - \Phi_{,\mu}n^{\mu}\Phi_{,\nu}n^{\nu}\right)$$
$$-\frac{8\pi G_o}{c^4}\frac{\bar{T}_{\mu\nu}}{\phi^2}\bar{n}^{\mu}\bar{n}^{\nu} = 0. \tag{4.37}$$

Use of the equation

$$\left(\frac{3}{2} + \omega\right)\left(\frac{1}{2}g_{\mu\nu}n^{\mu}n^{\nu}\Phi^{,\alpha}\Phi_{,\alpha} - \Phi_{,\mu}n^{\mu}\Phi_{,\nu}n^{\nu}\right) =$$
$$-\frac{1}{2}\frac{\phi^2\Pi^2h^{-1}}{(6+4\omega)} - \left(\frac{3}{2} + \omega\right)\frac{1}{2}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2} \tag{4.38}$$

will further modify equation (4.38) into the form

$$\frac{1}{2}[{}^{(3)}R + h^{-1}(\frac{\Pi^2}{2} - \Pi^{ab}\Pi_{ab})] - \frac{1}{2}\frac{\phi^2\Pi^2h^{-1}}{(6+4\omega)}$$
$$-\left(\frac{3}{2} + \omega\right)\frac{1}{2}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2} - \frac{8\pi G_o}{c^4}\frac{\bar{T}_{\mu\nu}}{\phi^2}\bar{n}^{\mu}\bar{n}^{\nu} = 0. \tag{4.39}$$

Finally, use of the transformed determinant

$$\bar{h}^{1/2} = \frac{h^{1/2}}{\phi^{3/2}} \tag{4.40}$$

will allow one to derive the Hamiltonian constraint

$$h^{1/2}[-^{(3)}R + h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2)] + \frac{\phi^2\Pi^2 h^{-1/2}}{(6+4\omega)} + \left(\frac{3}{2}+\omega\right)h^{1/2}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2}$$

$$= \frac{16\pi G_o}{c^4}\frac{\bar{h}^{1/2}}{\phi^{1/2}}\bar{T}_{ab}\bar{n}^a\bar{n}^b. \tag{4.41}$$

This expression is in agreement with equation (4.30). The momentum constraint can be derived in a similar, and straightforward, manner. First of all, one may calculate the functional derivative $\delta H_M/\delta N_d$, viz.

$$\frac{\delta H_M}{\delta N_d} = \frac{\bar{h}^{1/2}}{\phi}\int \frac{d^3\bar{p}d\bar{p}_t}{\bar{N}\bar{h}^{1/2}}\left[\frac{\bar{h}^{da}\bar{p}_a(\bar{\varepsilon}+\bar{N}^c\bar{p}_c)}{\bar{N}}\right]\frac{\bar{f}}{\bar{m}}. \tag{4.42}$$

The functional derivative of the gravitational contribution will be

$$\frac{\delta H_G}{\delta N_d} = \frac{c^4}{16\pi G_o}\{-2h^{1/2}D_a(h^{-1/2}\Pi^{ad}) + \Pi\phi^{,d}\}. \tag{4.43}$$

Combining equations (4.42) and (4.43) yields the momentum constraint

$$-2h^{1/2}D_a(h^{-1/2}\Pi^{ad}) + \Pi\phi^{,d}$$

$$= \frac{16\pi G_o}{c^4}\frac{\bar{h}^{1/2}}{\phi}\int \frac{d^3\bar{p}d\bar{p}_t}{\bar{N}\bar{h}^{1/2}}\left[\frac{\bar{h}^{da}\bar{p}_a(\bar{\varepsilon}+\bar{N}^c\bar{p}_c)}{\bar{N}}\right]\frac{\bar{f}}{\bar{m}}. \tag{4.44}$$

The momentum constraint (4.44) may be checked by taking the inner product of the Brans-Dicke field equation with the spatial metric $h^{\alpha\mu}$ and the unit normal vector n^μ. This will yield the following expression

$$h^{\alpha\mu}G_{\mu\nu}n^\nu + \left(\frac{3}{2}+\omega\right)h^{\alpha\mu}(\frac{1}{2}g_{\mu\nu}\Phi^{,\alpha}\Phi_{,\alpha} - \Phi_{,\mu}\Phi_{,\nu})n^\nu$$

$$-\frac{8\pi G_o}{c^4}h^{\alpha\mu}\frac{\bar{T}_{\mu\nu}}{\phi}n^\nu = 0. \tag{4.45}$$

The relations

$$h^{\alpha\mu}G_{\mu\nu}n^{\nu} = D_{\beta}(h^{-1/2}\Pi^{\beta\alpha}) \tag{4.46}$$

and

$$\left(\frac{3}{2}+\omega\right)h^{\alpha\mu}\left(\frac{1}{2}g_{\mu\nu}\Phi^{,\beta}\Phi_{,\beta} - \Phi_{,\mu}\Phi_{,\nu}\right)n^{\nu} = -\frac{1}{2}h^{-1/2}\Pi\phi^{,\alpha} \tag{4.47}$$

enable one to rewrite equation (4.45) in the form

$$D_{\beta}(h^{-1/2}\Pi^{\beta\alpha}) - \frac{1}{2}h^{-1/2}\Pi\phi^{,\alpha} - \frac{8\pi G_o}{c^4}h^{\alpha\mu}\frac{\bar{T}_{\mu\nu}}{\phi}n^{\nu} = 0. \tag{4.48}$$

Finally, one can calculate

$$\bar{h}^{\alpha\mu}\bar{T}_{\mu\nu}\bar{n}^{\nu} = -\int \frac{d^3\bar{p}d\bar{p}_t}{\bar{N}\bar{h}^{1/2}}\left[\bar{h}^{\alpha\mu}\bar{p}_{\mu}\left(\frac{\bar{\varepsilon}+\bar{N}^{\beta}\bar{p}_{\beta}}{\bar{N}}\right)\right]\frac{\bar{f}}{\bar{m}}, \tag{4.49}$$

which, when substituted into equation (4.48), yields the desired momentum constraint

$$-2h^{1/2}D_{\beta}(h^{-1/2}\Pi^{\beta\alpha}) + \Pi\phi^{,\alpha}$$

$$= \frac{16\pi G_o}{c^4}\frac{\bar{h}^{1/2}}{\phi}\int \frac{d^3\bar{p}d\bar{p}_t}{\bar{N}\bar{h}^{1/2}}\left[\bar{h}^{\alpha\mu}\bar{p}_{\mu}\left(\frac{\bar{\varepsilon}+\bar{N}^{\beta}\bar{p}_{\beta}}{\bar{N}}\right)\right]\frac{\bar{f}}{\bar{m}}. \tag{4.50}$$

The requirement that the correct dynamical equations be generated by the constraint equation

$$\partial_t F = \langle F, H\rangle, \tag{4.51}$$

which holds for arbitrary functionals F, implies that the Lie bracket and Hamiltonian combine to generate the Vlasov-Brans-Dicke system. Consequently, one may construct the appropriate dynamical equations

$$\partial_t h_{ab} = \langle h_{ab}, H\rangle = 16\pi\frac{\delta H}{\delta\Pi^{ab}}, \tag{4.52a}$$

$$\partial_t \phi = \langle \phi, H\rangle = 16\pi\frac{\delta H}{\delta\Pi}, \tag{4.52b}$$

$$\partial_t \Pi^{ab} = \langle \Pi^{ab}, H\rangle = -16\pi\frac{\delta H}{\delta h_{ab}} \tag{4.52c}$$

and

$$\partial_t \Pi = \langle \Pi, H \rangle = -16\pi \frac{\delta H}{\delta \phi}. \tag{4.52d}$$

Explicit forms for equations (4.52a) to (4.52d) may be derived. These are

$$\partial_t h_{ab} = 16\pi \frac{\delta H}{\delta \Pi^{ab}} = 2Nh^{-1/2}(\Pi_{ab} - \frac{1}{2}h_{ab}\Pi^c{}_c) + 2D_{(a}N_{b)}, \tag{4.53a}$$

$$\partial_t \phi = 16\pi \frac{\delta H}{\delta \Pi} = \frac{2Nh^{-1/2}\Pi\phi^2}{(6+4\omega)} + N^b\phi_{,b}, \tag{4.53b}$$

$$\begin{aligned}\partial_t \Pi = -16\pi \frac{\delta H}{\delta \phi} = &-\frac{2Nh^{-1/2}\Pi^2\phi}{(6+4\omega)} + 2Nh^{1/2}\left(\frac{3}{2}+\omega\right)\frac{D_bD^b\phi}{\phi^2} \\ &-2Nh^{1/2}\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^3} + 2h^{1/2}\left(\frac{3}{2}+\omega\right)\frac{\partial_b N\phi^{,b}}{\phi^2} \\ &+h^{1/2}D_a(N^a h^{-1/2}\Pi^c{}_c) + 32\pi\frac{Nh^{1/2}}{\phi^3}\bar{T}^t_t - 16\pi\frac{Nh^{1/2}}{\phi^2}\frac{\partial \bar{T}^t_t}{\partial \phi}\end{aligned} \tag{4.53c}$$

and

$$\begin{aligned}\partial_t \Pi^{ab} &= -16\pi \frac{\delta H}{\delta h_{ab}} \\ &= -Nh^{1/2}({}^{(3)}R^{ab} - \frac{1}{2}h^{ab(3)}R) + \frac{1}{2}Nh^{-1/2}h^{ab}(\Pi^{cd}\Pi_{cd} - \frac{1}{2}\Pi^2) \\ &-2Nh^{-1/2}(\Pi^{ac}\Pi_c{}^b - \frac{1}{2}\Pi\Pi^{ab}) + h^{1/2}(D^aD^bN - h^{ab}D_cD^cN) \\ &+\frac{1}{2}Nh^{-1/2}h^{ab}\frac{\phi^2\Pi^2}{(6+4\omega)} - \frac{1}{2}Nh^{1/2}h^{ab}\left(\frac{3}{2}+\omega\right)h^{cd}\frac{\phi_{,c}\phi_{,d}}{\phi^2} \\ &+\left(\frac{3}{2}+\omega\right)Nh^{1/2}h^{ca}h^{db}\frac{\phi_{,c}\phi_{,d}}{\phi^2} + h^{1/2}D_c(h^{-1/2}N^c\Pi^{ab}) \\ &-2\Pi^{c(a}D_cN^{b)} + 8\pi\frac{\bar{N}\bar{h}^{1/2}}{\phi}\int \frac{d^3\bar{p}d\bar{p}_t}{\bar{N}\bar{h}^{1/2}}\frac{\bar{f}}{\bar{m}}\bar{h}^{ca}\bar{p}_c\bar{p}_d\bar{h}^{db}.\end{aligned} \tag{4.53d}$$

Equation (4.53b) may be verified by computing the conjugate momentum $\Pi = \partial L/\partial\dot{\phi}$. Since we are working in the pseudo-Einstein representation of the Brans-Dicke theory, the equation (4.53a) for $\partial_t h_{ab}$ is what we would expect. To prove equation (4.53c) we must rewrite equation (4.10) as

$$(3+2\omega)Nh^{1/2}\frac{\nabla^\mu\nabla_\mu\Phi}{\phi} = \frac{32\pi G_o}{c^4}\frac{Nh^{1/2}}{\phi^3}\bar{L}_M - \frac{16\pi G_o}{c^4}\frac{Nh^{1/2}}{\phi^2}\frac{\delta\bar{L}_M}{\delta\phi}, \tag{4.54}$$

where

$$\Phi = \ln \phi. \tag{4.55}$$

We then proceed to write the D'Alembertian operator in the ADM formalism

$$\nabla^\mu \nabla_\mu \Phi = \frac{1}{\sqrt{|-g|}} \frac{\partial}{\partial x^\mu} (\sqrt{|-g|} g^{\mu\nu} \frac{\partial \Phi}{\partial x^\nu})$$
$$= \frac{1}{N\sqrt{h}} \frac{\partial}{\partial t} (-N\sqrt{h} \frac{1}{N^2} \Phi_{,t}) + \frac{1}{N\sqrt{h}} \frac{\partial}{\partial x^\alpha} (N\sqrt{h} \frac{N^\alpha}{N^2} \Phi_{,t})$$
$$+ \frac{1}{N\sqrt{h}} \frac{\partial}{\partial t} (N\sqrt{h} \frac{N^\alpha}{N^2} \Phi_{,\alpha}) + \frac{1}{N\sqrt{h}} \frac{\partial}{\partial x^\alpha} (N\sqrt{h} (h^{\alpha\beta} - \frac{N^\alpha N^\beta}{N^2}) \Phi_{,\beta}). \tag{4.56}$$

After a great deal of algebra, it is found that the dynamical equation $\partial_t \Pi$ (4.53c) is, indeed, equivalent to the scalar equation (4.10). In this derivation, one must use the result $\bar{L}_M = -\bar{T}^t_t$. In the limit of flat space-time, the dynamical equation (4.53c) reduces to the form of a wave equation for Φ.

A proof of equation (4.53d) may be obtained by computing the double inner product of equation (4.9) with h^{ac}, viz.

$$h^{ac} G_{cd} h^{db} = \frac{8\pi G_o}{c^4} h^{ac} \frac{\bar{T}_{cd}}{\phi} h^{db} + \left(\frac{3}{2} + \omega\right) h^{ac} \left(\Phi_{,c}\Phi_{,d} - \frac{1}{2} g_{cd} \Phi^{,e} \Phi_{,e}\right) h^{db}. \tag{4.57}$$

This expression may be reformulated as

$$N h^{1/2} G_{cd} h^{ac} h^{db} = \frac{1}{2} N h^{-1/2} \frac{h^{ab} \Pi^2 \phi^2}{(6+4\omega)} - \frac{1}{2}\left(\frac{3}{2} + \omega\right) N h^{1/2} h^{ab} h^{cd} \frac{\phi_{,c}\phi_{,d}}{\phi^2}$$
$$+ \left(\frac{3}{2} + \omega\right) N h^{1/2} \frac{h^{ac}\phi_{,c}\phi_{,d} h^{db}}{\phi^2} + \frac{8\pi G_o}{c^4} \frac{\bar{N}\bar{h}^{1/2}}{\phi} \int d\bar{\Gamma} \frac{\bar{f}}{\bar{m}} \bar{h}^{ac} \bar{p}_c \bar{p}_d \bar{h}^{db}, \tag{4.58}$$

which is equation (4.53d). The Vlasov equation possesses the form

$$\partial_t \bar{f} = \langle \bar{f}, H \rangle = 16\pi \int d^3x \left(\frac{\delta \bar{f}}{\delta \phi} \frac{\delta H}{\delta \Pi}\right) + \int d\bar{\Gamma} \bar{f} \left\{ \frac{\delta \bar{f}}{\delta \bar{f}}, \frac{\delta H}{\delta \bar{f}} \right\}_{\bar{x},\bar{p}}$$
$$= 16\pi \frac{\partial \bar{f}}{\partial \phi} \frac{\delta H}{\delta \Pi} + \{\bar{\varepsilon}, \bar{f}\}_{\bar{x},\bar{p}}$$
$$= \{\bar{\varepsilon}, \bar{f}\}_{\bar{x},\bar{p}} + \frac{\partial \bar{f}}{\partial \phi} \partial_t \phi, \tag{4.59}$$

where equation (4.53b) has been used in the derivation. Consequently, one can write equation (4.59) as

$$\partial_t \bar{f} = \frac{\partial \bar{\varepsilon}}{\partial \bar{x}^a}\frac{\partial \bar{f}}{\partial \bar{p}_a} - \frac{\partial \bar{\varepsilon}}{\partial \bar{p}_a}\frac{\partial \bar{f}}{\partial \bar{x}^a} + \frac{\partial \bar{f}}{\partial \phi}\partial_t \phi \tag{4.60}$$

or as

$$\partial_t \bar{f} = \frac{\partial \varepsilon}{\partial x^a}\frac{\partial \bar{f}}{\partial p_a} - \frac{\partial \varepsilon}{\partial p_a}\frac{\partial \bar{f}}{\partial x^a} + \frac{\partial \bar{f}}{\partial \phi}\partial_t \phi, \tag{4.61}$$

where $\bar{\varepsilon} = \varepsilon$, $\bar{x}^a = x^a$ and $\bar{p}_a = p_a$. Substitution of

$$\frac{\partial \varepsilon}{\partial x^a} = \frac{\partial g^{\mu\nu}}{\partial x^a}\frac{p_\mu p_\nu}{2p^t} \tag{4.62a}$$

and

$$\frac{\partial \varepsilon}{\partial p_c} = \frac{p^c}{p^t} \tag{4.62b}$$

into equation (4.61) yields the result

$$\partial_t \bar{f} = -\frac{p^a}{p^t}\frac{\partial \bar{f}}{\partial x^a} + \frac{\partial g^{\mu\nu}}{\partial x^a}\frac{p_\mu p_\nu}{2p^t}\frac{\partial \bar{f}}{\partial p_a} + \frac{\partial \bar{f}}{\partial \phi}\partial_t \phi. \tag{4.63}$$

From Bartholomew (1971), we know that any phase-preserving perturbation can be written in the form

$$\delta \bar{f} = \exp(\{\bar{g}, .\})\bar{f}_o = \{\bar{g}, \bar{f}_o\} + \frac{1}{2}\{\bar{g}, \{\bar{g}, \bar{f}_o\}\} + \ldots \ . \tag{4.64}$$

We know that generic time-independent equilibria, $\{h^o{}_{ab}, \Pi_o{}^{ab}; \phi^o, \Pi_o; \bar{f}_o\}$, are energy extrema with respect to dynamically accessible perturbations,

$$\{\delta h_{ab}, \delta \Pi^{ab}; \delta\phi, \delta\Pi; \delta \bar{f}\}, \tag{4.65}$$

which satisfy

$$C[\bar{f} + \delta \bar{f}] - C[\bar{f}] = 0, \tag{4.66}$$

where

$$C[\bar{f}] = \int d\bar{\Gamma} \chi(\bar{f}). \tag{4.67}$$

For a phase-preserving perturbation $\delta^{(1)}\bar{f}$, we can prove that the first variation $\delta^{(1)}H$ vanishes identically. We calculate the first order variation of the Hamiltonian

$$\delta^{(1)}H = \int d^3x\Big(\frac{\delta\mathcal{H}}{\delta h_{ab}}\delta^{(1)}h_{ab} + \frac{\delta\mathcal{H}}{\delta\Pi^{ab}}\delta^{(1)}\Pi^{ab} + \frac{\delta\mathcal{H}}{\delta\phi}\delta^{(1)}\phi + \frac{\delta\mathcal{H}}{\delta\Pi}\delta^{(1)}\Pi$$

$$+\frac{\delta\mathcal{H}}{\delta N}\delta^{(1)}N + \frac{\delta\mathcal{H}}{\delta N_a}\delta^{(1)}N_a + \frac{\delta\mathcal{H}}{\delta\bar{f}}\delta^{(1)}\bar{f}\Big), \tag{4.68}$$

which reduces to

$$\delta^{(1)}H = \frac{1}{16\pi}\int d^3x\Big(-\partial_t\Pi^{ab}\delta^{(1)}h_{ab} + \partial_t h_{ab}\delta^{(1)}\Pi^{ab} - \partial_t\Pi\delta^{(1)}\phi + \partial_t\phi\delta^{(1)}\Pi\Big)$$

$$+\int d\bar{\Gamma}\bar{\varepsilon}_o\delta^{(1)}\bar{f}. \tag{4.69}$$

For a time-independent equilibrium this expression reduces to

$$\delta^{(1)}H = \int d\bar{\Gamma}\bar{\varepsilon}_o\delta^{(1)}\bar{f} = \int d\bar{\Gamma}\bar{\varepsilon}_o\{\bar{g},\bar{f}_o\} = -\int d\bar{\Gamma}\bar{g}\{\bar{\varepsilon}_o,\bar{f}_o\} = 0, \tag{4.70}$$

because

$$\{\bar{\varepsilon}_o,\bar{f}_o\} = \partial_t\bar{f} - \frac{\partial\bar{f}}{\partial\phi}\partial_t\phi = 0 \tag{4.71}$$

for a static equilibrium.

4.2 The Second Variation $\delta^{(2)}H$

Following Kandrup and O'Neill (1994), let Y^I represent the set of variables

$$\{h_{ab},\Pi^{ab},\Phi,\Pi,N,N_a,\bar{f}\}. \tag{4.72}$$

The second variation of the Hamiltonian, $\delta^{(2)}H$, can be written

$$\delta^{(2)}H = \sum_I\frac{\delta H}{\delta Y^I}\delta^{(2)}Y^I + \frac{1}{2}\sum_I\sum_J\frac{\delta^{(2)}H}{\delta Y^I\delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J, \tag{4.73}$$

where the functional derivatives are evaluated at the unperturbed equilibrium. As in Kandrup and O'Neill (1994), the equations of constraint and the dynamical equations, evaluated

at an unperturbed equilibrium, reduce the first term of equation (4.73) to the single contribution

$$\sum_I \frac{\delta H}{\delta Y^I}\delta^{(2)}Y^I = \int d\bar{\Gamma}\bar{\varepsilon}_o\delta^{(2)}\bar{f}. \tag{4.74}$$

Therefore, one may write $\delta^{(2)}H$ as

$$\delta^{(2)}H = -\frac{1}{2}\int d\bar{\Gamma}\{\bar{g},\bar{\varepsilon}_o\}\{\bar{g},\bar{f}_o\} + \frac{1}{2}\sum_I\sum_J\int d\bar{\Gamma}\frac{\partial^{(2)}(\bar{f}\bar{\varepsilon})}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J$$
$$+\frac{1}{2}\sum_I\sum_J\frac{\delta^{(2)}H_G}{\delta Y^I\delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J, \tag{4.75}$$

where the second order phase-preserving perturbation $\delta^{(2)}\bar{f} = \frac{1}{2}\{\bar{g},\{\bar{g},\bar{f}_o\}\}$, given in equation (4.64), has been used in the derivation. Following the calculation given in chapter 3 for the Vlasov-Einstein system (Kandrup and O'Neill (1994)), we find that

$$\delta^{(2)}H = -\frac{1}{2}\int d\bar{\Gamma}\{\bar{g},\bar{\varepsilon}_o\}\{\bar{g},\bar{f}_o\} + \sum_I\int d\bar{\Gamma}\delta^{(1)}Y^I\delta^{(1)}\left(\bar{f}\frac{\partial\bar{\varepsilon}}{\partial Y^I}\right)$$
$$-\frac{1}{2}\sum_I\sum_J\int d\bar{\Gamma}\bar{f}\frac{\partial^{(2)}\bar{\varepsilon}}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J$$
$$+\frac{1}{2}\sum_I\sum_J\frac{\delta^{(2)}H_G}{\delta Y^I\delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J. \tag{4.76}$$

Due to the fact that $\bar{\varepsilon} = \varepsilon$, one can rewrite the expression

$$\sum_I\int d\bar{\Gamma}\delta^{(1)}Y^I\delta^{(1)}\left(\bar{f}\frac{\partial\bar{\varepsilon}}{\partial Y^I}\right) \tag{4.77a}$$

as

$$\sum_I\int d\bar{\Gamma}\delta^{(1)}Y^I\delta^{(1)}\left(\bar{f}\frac{\partial\varepsilon}{\partial Y^I}\right). \tag{4.77b}$$

From Kandrup and O'Neill (1994), we know that

$$\delta^{(1)}d\Gamma = 0, \tag{4.78}$$

and, since $d\bar{\Gamma} = \phi^{1/2}d\Gamma = \phi^{1/2}d^3xd^3pdm$, one can calculate the variation of the transformed phase space volume element

$$\delta^{(1)}d\bar{\Gamma} = \frac{1}{2}\phi^{-1/2}\delta^{(1)}\phi d\Gamma + \phi^{1/2}\delta^{(1)}d\Gamma = \frac{1}{2}\frac{\delta^{(1)}\phi}{\phi}d\bar{\Gamma}. \tag{4.79}$$

Expanding the summation of equation (4.77b) yields the result

$$\int d\bar{\Gamma}\delta^{(1)}N\delta^{(1)}\Big(\bar{f}\frac{\partial\varepsilon}{\partial N}\Big) + \int d\bar{\Gamma}\delta^{(1)}N_a\delta^{(1)}\Big(\bar{f}\frac{\partial\varepsilon}{\partial N_a}\Big) + \int d\bar{\Gamma}\delta^{(1)}h_{ab}\delta^{(1)}\Big(\bar{f}\frac{\partial\varepsilon}{\partial h_{ab}}\Big). \tag{4.80}$$

This can be written as

$$\int d^3x\Big[\delta^{(1)}\bar{\rho}\delta^{(1)}N - \frac{\bar{\rho}}{2\phi}\delta^{(1)}N\delta^{(1)}\phi + \delta^{(1)}\bar{J}^a\delta^{(1)}N_a - \frac{\bar{J}^a}{2\phi}\delta^{(1)}N_a\delta^{(1)}\phi$$
$$+\frac{1}{2}\delta^{(1)}(N\bar{S}^{ab} - 2N^a\bar{J}^b)\delta^{(1)}h_{ab} - \frac{1}{4}(N\bar{S}^{ab} - 2N^a\bar{J}^b)\delta^{(1)}h_{ab}\frac{\delta^{(1)}\phi}{\phi}\Big], \tag{4.81}$$

where

$$\bar{\rho} = \frac{\bar{h}^{1/2}}{\phi^{1/2}}\bar{n}^a\bar{T}_{ab}\bar{n}^b, \quad \bar{J}^a = \frac{\bar{h}^{1/2}}{\phi}\bar{h}^{ac}\bar{T}_{cb}\bar{n}^b,$$

and

$$\bar{S}^{ab} = \frac{\bar{h}^{1/2}}{\phi^{3/2}}\bar{h}^{ac}\bar{T}_{cd}\bar{h}^{db}.$$

As in chapter 3, the first order variations $\delta\bar{\rho}$, $\delta\bar{J}^a$, and $\delta\bar{S}^{ab}$ are given in terms of a perturbed distribution function $\delta^{(1)}\bar{f} = \{\bar{g}, \bar{f}_o\}$. We, now, wish to expand the term

$$-\frac{1}{2}\sum_I\sum_J\int d\bar{\Gamma}\bar{f}\frac{\partial^2\bar{\varepsilon}}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J =$$
$$-\frac{1}{2}\sum_I\sum_J\int d\bar{\Gamma}\bar{f}\frac{\partial^2\varepsilon}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J, \tag{4.82}$$

which yields the following expression

$$-\int d\bar{\Gamma}\bar{f}\frac{\partial^2\varepsilon}{\partial h_{ab}\partial N}\delta^{(1)}h_{ab}\delta^{(1)}N - \int d\bar{\Gamma}\bar{f}\frac{\partial^2\varepsilon}{\partial h_{ab}\partial N_c}\delta^{(1)}h_{ab}\delta^{(1)}N_c$$
$$-\frac{1}{2}\int d\bar{\Gamma}\bar{f}\frac{\partial^2\varepsilon}{\partial h_{ab}\partial h_{cd}}\delta^{(1)}h_{ab}\delta^{(1)}h_{cd}. \tag{4.83}$$

In the above derivation, one uses the fact that

$$\frac{\partial^2\varepsilon}{\partial N^2} = \frac{\partial^2\varepsilon}{\partial N\partial N_c} = \frac{\partial^2\varepsilon}{\partial N_a\partial N_b} = 0. \tag{4.84}$$

One may compute equation (4.83) to be of the form

$$\int d^3x\Big[-\frac{1}{2}\bar{S}^{ab}\delta^{(1)}h_{ab}\delta^{(1)}N + \bar{J}^b h^{ac}\delta^{(1)}h_{ab}\delta^{(1)}N_c$$
$$+\Big(\frac{1}{8}N\bar{S}^{abcd} - \frac{1}{2}Nh^{ac}\bar{S}^{bd} - h^{ac}\bar{J}^b N^d\Big)\delta^{(1)}h_{ab}\delta^{(1)}h_{cd}\Big], \tag{4.85}$$

where

$$\bar{S}^{abcd} = \frac{\bar{h}^{1/2}}{\phi^{5/2}}\bar{h}^{ae}\bar{h}^{bf}\bar{h}^{cg}\bar{h}^{dh}\int \frac{d^4\bar{p}}{\sqrt{-\bar{g}}}\frac{\bar{f}}{\bar{m}}\frac{\bar{p}_e\bar{p}_f\bar{p}_g\bar{p}_h}{(\bar{n}^k\bar{p}_k)^2}. \tag{4.86}$$

A combination of equations (4.81) and (4.85) yields the final result

$$\int d^3x\Big[\delta^{(1)}\bar{\rho}\delta^{(1)}N - \frac{\bar{\rho}}{2\phi}\delta^{(1)}N\delta^{(1)}\phi + \delta^{(1)}\bar{J}^a\delta^{(1)}N_a - \frac{\bar{J}^a}{2\phi}\delta^{(1)}N_a\delta^{(1)}\phi$$
$$+\frac{1}{2}\delta^{(1)}(N\bar{S}^{ab} - 2N^a\bar{J}^b)\delta^{(1)}h_{ab} - \frac{1}{4}(N\bar{S}^{ab} - 2N^a\bar{J}^b)\delta^{(1)}h_{ab}\frac{\delta^{(1)}\phi}{\phi}$$
$$-\frac{1}{2}\bar{S}^{ab}\delta^{(1)}h_{ab}\delta^{(1)}N + \bar{J}^b h^{ac}\delta^{(1)}h_{ab}\delta^{(1)}N_c$$
$$+\Big(\frac{1}{8}N\bar{S}^{abcd} - \frac{1}{2}Nh^{ac}\bar{S}^{bd} - h^{ac}\bar{J}^b N^d\Big)\delta^{(1)}h_{ab}\delta^{(1)}h_{cd}\Big]. \tag{4.87}$$

This expression may be further reduced through the use of the identities

$$\delta^{(1)}(N^a\bar{J}^b) = \bar{J}^b h^{ac}\delta^{(1)}N_c + N^a\delta^{(1)}\bar{J}^b - h^{ac}\bar{J}^b N^d\delta^{(1)}h_{cd} \tag{4.88a}$$

and

$$\delta^{(1)}(N\bar{S}^{ab}) = \bar{S}^{ab}\delta^{(1)}N + N\delta^{(1)}\bar{S}^{ab}. \tag{4.88b}$$

We can write the expression for $\delta^{(2)}H$ as

$$\delta^{(2)}H = -\frac{1}{2}\int d\bar{\Gamma}\{\bar{g},\bar{\varepsilon}_o\}\{\bar{g},\bar{f}_o\} + \int d^3x\Big[\delta^{(1)}\bar{\rho}\delta^{(1)}N - \frac{\bar{\rho}}{2\phi}\delta^{(1)}N\delta^{(1)}\phi$$
$$+\delta^{(1)}\bar{J}^a\delta^{(1)}N_a - \frac{\bar{J}^a}{2\phi}\delta^{(1)}N_a\delta^{(1)}\phi + \frac{1}{2}N\delta^{(1)}\bar{S}^{ab}\delta^{(1)}h_{ab}$$
$$-N^a\delta^{(1)}\bar{J}^b\delta^{(1)}h_{ab} - \frac{1}{4}(N\bar{S}^{ab} - 2N^a\bar{J}^b)\delta^{(1)}h_{ab}\frac{\delta^{(1)}\phi}{\phi}$$
$$+(\frac{1}{8}N\bar{S}^{abcd} - \frac{1}{2}Nh^{ac}\bar{S}^{bd})\delta^{(1)}h_{ab}\delta^{(1)}h_{cd}\Big]$$
$$+\delta^{(2)}H_G[\delta^{(1)}Y], \tag{4.89}$$

where

$$\delta^{(2)}H_G[\delta^{(1)}Y] = +\frac{1}{2}\sum_I\sum_J \frac{\delta^{(2)}H_G}{\delta Y^I \delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J \tag{4.90}$$

represents the second order energy associated with a first order perturbation $\delta^{(1)}Y$. A straightforward calculation of the gravitational contribution yields

$$\begin{aligned}16\pi\delta^{(2)}H_G[\delta^{(1)}Y] = \int d^3x\{\delta^{(1)}N[h^{1/2}({}^{(3)}R^{ab} - \frac{1}{2}h^{ab(3)}R)\delta^{(1)}h_{ab}\\
+h^{1/2}(h^{ab}D_cD^c\delta^{(1)}h_{ab} - D^aD^b\delta^{(1)}h_{ab})\\
-\frac{1}{2}h^{-1/2}h^{ab}\delta^{(1)}h_{ab}(\Pi^{cd}\Pi_{cd} - \frac{1}{2}\Pi^2)\\
+2h^{-1/2}(\Pi_{ab} - \frac{1}{2}h_{ab}\Pi)\delta^{(1)}\Pi^{ab} + 2h^{-1/2}(\Pi^{ac}\Pi^b{}_c - \Pi\Pi^{ab})\delta^{(1)}h_{ab}\\
-\frac{1}{2}h^{-1/2}h^{ab}\delta^{(1)}h_{ab}\frac{\phi^2\Pi^2}{(6+4\omega)} + 2h^{-1/2}\frac{\phi\delta^{(1)}\phi\Pi^2}{(6+4\omega)} + 2h^{-1/2}\frac{\Pi\delta^{(1)}\Pi\phi^2}{(6+4\omega)}\\
+\left(\frac{3}{2}+\omega\right)h^{1/2}h^{ab}\delta^{(1)}h_{ab}h^{cd}\frac{\phi_{,c}\phi_{,d}}{2\phi^2} + \left(\frac{3}{2}+\omega\right)h^{1/2}\delta^{(1)}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2}\\
+2\left(\frac{3}{2}+\omega\right)h^{1/2}h^{ab}\frac{\delta^{(1)}\phi_{,a}\phi_{,b}}{\phi^2} - 2\left(\frac{3}{2}+\omega\right)h^{1/2}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^3}\delta^{(1)}\phi]\}\\
+\int d^3x\{h^{1/2}D_c[h^{-1/2}(2\Pi^{ac}\delta^{(1)}N^b - \Pi^{ab}\delta^{(1)}N^c)]\delta^{(1)}h_{ab}\\
+2\delta^{(1)}\Pi^{ab}D_a\delta^{(1)}N_b\\
+\delta^{(1)}N_b\delta^{(1)}\Pi\phi^{,b} + \delta^{(1)}N_b\Pi\delta^{(1)}h^{ba}\phi_{,a} + \delta^{(1)}N_b\Pi\delta^{(1)}\phi^{,b}\}\\
+\frac{1}{2}\int d^3x(ND^2H - N_aD^2\Delta^a).\end{aligned} \tag{4.91}$$

We may compute D^2H and $D^2\Delta^a$. One may write D^2H as the sum of two terms D^2H_1 and D^2H_2. The first contribution to D^2H (depending solely on metric terms) is given by

$$\begin{aligned}D^2H_1 = h^{-1/2}\{[\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2][\frac{1}{4}(\delta^{(1)}h^c{}_c)^2 + \frac{1}{2}\delta^{(1)}h^{cd}\delta^{(1)}h_{cd}]\\
-2\Pi^{ab}\Pi^{cd}(h_{bd}\delta^{(1)}h_{ac} - \frac{1}{2}h_{cd}\delta^{(1)}h_{ab})\delta^{(1)}h^e{}_e - 2(\Pi_{ab}\delta^{(1)}\Pi^{ab} - \frac{1}{2}\Pi\delta^{(1)}\Pi)\delta^{(1)}h^c{}_c\\
+2\Pi^{ab}\Pi^{cd}(\delta^{(1)}h_{ac}\delta^{(1)}h_{bd} - \frac{1}{2}\delta^{(1)}h_{ab}\delta^{(1)}h_{cd})\end{aligned}$$

$$+2\delta^{(1)}\Pi^{ab}\delta^{(1)}\Pi^{cd}(h_{ac}h_{bd}-\frac{1}{2}h_{ab}h_{cd})$$

$$+2(h_{bd}\delta^{(1)}h_{ac}+h_{ac}\delta^{(1)}h_{bd}-\frac{1}{2}h_{cd}\delta^{(1)}h_{ab}-\frac{1}{2}h_{ab}\delta^{(1)}h_{cd})$$

$$\times(\Pi^{cd}\delta^{(1)}\Pi^{ab}+\Pi^{ab}\delta^{(1)}\Pi^{cd})\}$$

$$-h^{1/2}\{^{(3)}R\left[\frac{1}{4}(\delta^{(1)}h^{a}{}_{a})^{2}-\frac{1}{2}\delta^{(1)}h^{ab}\delta^{(1)}h_{ab}\right]-{}^{(3)}R_{ab}\delta^{(1)}h^{ab}\delta^{(1)}h^{c}{}_{c}$$

$$+2^{(3)}R_{ab}\delta^{(1)}h^{ac}\delta^{(1)}h_{c}{}^{b}$$

$$-\delta^{(1)}h^{ab}(D^{c}D_{b}\delta^{(1)}h_{ac}+D^{c}D_{a}\delta^{(1)}h_{bc})+\delta^{(1)}h^{c}{}_{c}(D^{b}D^{a}\delta^{(1)}h_{ab}-D^{b}D_{b}\delta^{(1)}h^{a}{}_{a})$$

$$+2\delta^{(1)}h^{cd}(D^{a}D_{a}\delta^{(1)}h_{cd}+D_{d}D_{c}\delta^{(1)}h^{a}{}_{a}+D_{d}D^{a}\delta^{(1)}h_{ac})$$

$$+[\frac{1}{2}D^{c}\delta^{(1)}h^{b}{}_{b}-D_{b}\delta^{(1)}h^{bc}](2D^{a}\delta^{(1)}h_{ac}-D_{c}\delta^{(1)}h^{a}{}_{a})$$

$$+\frac{1}{2}D_{a}\delta^{(1)}h_{cd}D^{a}\delta^{(1)}h^{cd}+D^{d}\delta^{(1)}h_{a}{}^{c}(D_{d}\delta^{(1)}h_{c}{}^{a}-D_{c}\delta^{(1)}h_{d}{}^{a})\}. \qquad (4.92a)$$

The second contribution to D^2H (with dependence on scalar terms) is given by:

$$D^{2}H_{2}=-h^{1/2}\{-2\left[\frac{1}{4}(\delta^{(1)}h^{e}{}_{e})^{2}-\frac{1}{2}\delta^{(1)}h^{cd}\delta^{(1)}h_{cd}\right]\frac{h^{-1}\phi^{2}\Pi^{2}}{(6+4\omega)}$$

$$+2h^{-1}\delta^{(1)}h^{e}{}_{e}\left[\frac{2\phi\Pi^{2}\delta^{(1)}\phi}{(6+4\omega)}+\frac{2\Pi\phi^{2}\delta^{(1)}\Pi}{(6+4\omega)}\right]$$

$$-2h^{-1}\left[\frac{2(\delta^{(1)}\phi^{2})\Pi^{2}}{(6+4\omega)}+\frac{2\phi^{2}(\delta^{(1)}\Pi)^{2}}{(6+4\omega)}+\frac{8\phi\Pi\delta^{(1)}\phi\delta^{(1)}\Pi}{(6+4\omega)}\right]$$

$$-2\left[\frac{1}{4}(\delta^{(1)}h^{e}{}_{e})^{2}+\frac{1}{2}\delta^{(1)}h^{cd}\delta^{(1)}h_{cd}\right]\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^{2}}$$

$$-2\delta^{(1)}h^{e}{}_{e}\left[\left(\frac{3}{2}+\omega\right)\delta^{(1)}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^{2}}+2\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\delta^{(1)}\phi_{,a}\phi_{,b}}{\phi^{2}}-2\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^{3}}\delta^{(1)}\phi\right]$$

$$-2\delta^{(1)}h^{ab}\left[4\left(\frac{3}{2}+\omega\right)\frac{\delta^{(1)}\phi_{,a}\phi_{,b}}{\phi^{2}}-4\left(\frac{3}{2}+\omega\right)\frac{\phi_{,a}\phi_{,b}}{\phi^{3}}\delta^{(1)}\phi\right]$$

$$-4\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\delta^{(1)}\phi_{,a}\delta^{(1)}\phi_{,b}}{\phi^{2}}+16\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\delta^{(1)}\phi_{,a}\phi_{,b}}{\phi^{3}}\delta^{(1)}\phi$$

$$-12\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^{4}}(\delta^{(1)}\phi^{2})\}. \qquad (4.92b)$$

The expression for $D^2\Delta^a$ is given as

$$D^{2}\Delta^{a}=2(\delta^{(1)}\Pi^{bc}h^{ad}-\Pi^{bc}\delta^{(1)}h^{ad})$$

$$\times(D_c\delta^{(1)}h_{bd} + D_b\delta^{(1)}h_{cd} - D_d\delta^{(1)}h_{bc})$$

$$-4\delta^{(1)}\Pi\delta^{(1)}h^{ab}\phi_{,b} - 4(\delta^{(1)}\Pi h^{ab} + \Pi\delta^{(1)}h^{ab})\delta^{(1)}\phi_{,b}. \qquad (4.92c)$$

The energy functional may be reduced to a simpler format by using the perturbed constraint equation. The Hamiltonian constraint can be written as

$$^{(3)}R + h^{-1}(\frac{1}{2}\Pi^2 - \Pi^{ab}\Pi_{ab}) - \frac{h^{-1}\phi^2\Pi^2}{(6+4\omega)} - \left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2} = \frac{16\pi G_o}{c^4}h^{-1/2}\bar{\rho}, \qquad (4.93a)$$

where

$$\bar{\rho} = \frac{\bar{h}^{1/2}}{\phi^{1/2}}\bar{T}_{ab}\bar{n}^a\bar{n}^b. \qquad (4.93b)$$

One can compute the perturbed form of equation (4.93a) to be

$$-\delta^{(1)}h_{ab}{}^{(3)}R^{ab} - D_cD^c\delta^{(1)}h_d{}^d + D^aD^b\delta^{(1)}h_{ab}$$

$$+h^{-1}h^{cd}\delta^{(1)}h_{cd}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2)$$

$$-2h^{-1}\delta^{(1)}\Pi^{ab}(\Pi_{ab} - \frac{1}{2}h_{ab}\Pi) - 2h^{-1}\delta^{(1)}h_{ab}(\Pi^b{}_c\Pi^{ac} - \frac{1}{2}\Pi\Pi^{ab})$$

$$+h^{-1}h^{cd}\delta^{(1)}h_{cd}\frac{\phi^2\Pi^2}{(6+4\omega)} - 2h^{-1}\frac{\phi\delta^{(1)}\phi\Pi^2}{(6+4\omega)} - 2h^{-1}\frac{\phi^2\Pi\delta^{(1)}\Pi}{(6+4\omega)}$$

$$-\left(\frac{3}{2}+\omega\right)\delta^{(1)}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2} - 2\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\delta^{(1)}\phi_{,a}\phi_{,b}}{\phi^2} + 2\left(\frac{3}{2}+\omega\right)h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^3}\delta^{(1)}\phi$$

$$= \frac{16\pi G_o}{c^4}h^{-1/2}\delta^{(1)}\bar{\rho} - \frac{8\pi G_o}{c^4}h^{-1/2}\bar{\rho}h^{ab}\delta^{(1)}h_{ab}. \qquad (4.94)$$

The use of the unperturbed constraint (4.93a) allows one to derive the expression

$$h^{1/2}\left(\frac{1}{2}h^{ab(3)}R - {}^{(3)}R^{ab}\right)\delta^{(1)}h_{ab} + h^{1/2}(D^aD^b\delta^{(1)}h_{ab} - h^{ab}D_cD^c\delta^{(1)}h_{ab})$$

$$+\frac{1}{2}\delta^{(1)}h_{cd}h^{-1/2}h^{cd}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) - 2h^{-1/2}\delta^{(1)}\Pi^{ab}(\Pi_{ab} - \frac{1}{2}h_{ab}\Pi)$$

$$-2h^{-1/2}\delta^{(1)}h_{ab}(\Pi^b{}_c\Pi^{ac} - \frac{1}{2}\Pi\Pi^{ab}) + \frac{1}{2}h^{-1/2}h^{cd}\delta^{(1)}h_{cd}\frac{\phi^2\Pi^2}{(6+4\omega)}$$

$$-2h^{-1/2}\frac{\phi\delta^{(1)}\phi\Pi^2}{(6+4\omega)} - 2h^{-1/2}\frac{\phi^2\Pi\delta^{(1)}\Pi}{(6+4\omega)} - \left(\frac{3}{2}+\omega\right)h^{1/2}\delta^{(1)}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^2}$$

$$-2\left(\frac{3}{2}+\omega\right)h^{1/2}h^{ab}\frac{\delta^{(1)}\phi_{,a}\phi_{,b}}{\phi^2}+2\left(\frac{3}{2}+\omega\right)h^{1/2}h^{ab}\frac{\phi_{,a}\phi_{,b}}{\phi^3}\delta^{(1)}\phi$$

$$-\left(\frac{3}{2}+\omega\right)h^{1/2}h^{cd}\frac{\phi_{,c}\phi_{,d}}{2\phi^2}h^{ab}\delta^{(1)}h_{ab}=\frac{16\pi G_o}{c^4}\delta^{(1)}\bar{\rho}. \tag{4.95}$$

One may carry out the same procedure for the momentum constraint

$$D_a(h^{-1/2}\Pi^{ad})-\frac{1}{2}\Pi\phi^{,d}h^{-1/2}=\frac{8\pi G_o}{c^4}h^{-1/2}\bar{J}^d,$$

which has a perturbed form

$$2D_a\delta^{(1)}\Pi^{ab}+\Pi^{ac}h^{bd}(2D_c\delta^{(1)}h_{da}-D_d\delta^{(1)}h_{ac})$$

$$-\delta^{(1)}\Pi\phi^{,b}-\Pi\delta^{(1)}h^{bc}\phi_{,c}-\Pi\delta^{(1)}\phi^{,b}=\frac{16\pi G_o}{c^4}\delta^{(1)}\bar{J}^b. \tag{4.96}$$

The utilization of equations (4.95) and (4.96) allows one to rewrite equation (4.91) in the form

$$\delta^{(2)}H_G[\delta^{(1)}Y]=-\int d^3x(\delta^{(1)}\bar{\rho}\delta^{(1)}N+\delta^{(1)}\bar{J}^a\delta^{(1)}N_a)$$

$$+\frac{1}{32\pi}\int d^3x(ND^2H-N_aD^2\Delta^a). \tag{4.97}$$

Consequently, the expression for $\delta^{(2)}H$ (4.89) can be written as

$$\delta^{(2)}H=-\frac{1}{2}\int d\bar{\Gamma}\{\bar{g},\bar{\varepsilon}_o\}\{\bar{g},\bar{f}_o\}+\int d^3x\Big[-\frac{\bar{\rho}}{2\phi}\delta^{(1)}N\delta^{(1)}\phi$$

$$-\frac{\bar{J}^a}{2\phi}\delta^{(1)}N_a\delta^{(1)}\phi+\frac{1}{2}N\delta^{(1)}\bar{S}^{ab}\delta^{(1)}h_{ab}$$

$$-N^a\delta^{(1)}\bar{J}^b\delta^{(1)}h_{ab}-\frac{1}{4}(N\bar{S}^{ab}-2N^a\bar{J}^b)\delta^{(1)}h_{ab}\frac{\delta^{(1)}\phi}{\phi}$$

$$+\left(\frac{1}{8}N\bar{S}^{abcd}-\frac{1}{2}Nh^{ac}\bar{S}^{bd}\right)\delta^{(1)}h_{ab}\delta^{(1)}h_{cd}\Big]$$

$$+\frac{1}{64\pi}\int d^3x(ND^2H-N_aD^2\Delta^a). \tag{4.98}$$

4.3 The Question of Stability

It has been shown by Reddy (1973) that a Birkhoff theorem is not generally satisfied by a spherically symmetric matter configuration in the Brans-Dicke (1961) gravitational theory. Therefore, it is possible that gravitational radiation may escape from a pulsating spherically symmetric source. If, in the case of a spherical matter configuration, a negative second order perturbation of the Hamiltonian $\delta^{(2)}H < 0$ is permitted, will the system tend towards equilibrium or will it possess a run-away instability? We may answer this question by determining the sign of the time derivative of the second order Hamiltonian perturbation

$$\frac{d\delta^{(2)}H}{dt}. \tag{4.99}$$

To determine the sign of equation (4.99) one may exploit a Hamiltonian series expansion about equilibrium of the type given by Bruckman and Velázquez (1993), viz.

$$H = H_o + \delta^{(1)}H + \frac{1}{2}\delta^{(2)}H + ... \quad . \tag{4.100}$$

The condition of equilibrium, $\delta^{(1)}H = 0$, allows one to rewrite the equation (4.100) in the format

$$H = H_o + \frac{1}{2}\delta^{(2)}H + ... \quad . \tag{4.101}$$

The following expression

$$\frac{dE}{dt} = \frac{1}{2}\frac{d\delta^{(2)}H}{dt} \tag{4.102}$$

may be deduced since H_o is a constant and $H = E$. Determining the sign of dE/dt is equivalent to finding the sign of equation (4.99). We know from Wagoner (1970) that the simplest situation in which pure scalar radiation is produced remains the radial pulsation of a spherically symmetric matter configuration.

Following Wagoner (1970) one may deduce that the scalar-energy loss from such a configuration will be

$$-\left(\frac{dE}{dt}\right)_s = 4\pi {R_o}^2 c {T_s}^{tt}, \tag{4.103}$$

where the s subscript denotes scalar, R_o denotes a suitable far-field radial coordinate, c represents the speed of light, and ${T_s}^{tt}$ corresponds to the time-time component of the stress-energy tensor associated with the scalar field. The equation (4.103) holds for the variable G representation.

Intuitively, this expression is intrinsically negative because scalar monopole radiation, from a radially pulsating spherical source, will carry away energy, rendering the right side of equation (4.103) positive definite. Equation (4.103) can be written in terms of the variable mass representation as

$$-\left(\frac{d\bar{E}}{d\bar{t}}\right)_s = 4\pi \bar{R}_o^2 \bar{c} \bar{T}_s^{tt} = 4\pi {R_o}^2 c \bar{T}_s^{tt}. \tag{4.104}$$

The variable mass form for the stress-energy tensor associated with the scalar field is found, by inspection of equation (4.9), to be

$$\bar{T}^s_{\mu\nu} = \frac{c^4}{8\pi G^*}(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\Phi_{,\beta}\Phi_{,\alpha}), \tag{4.105a}$$

where

$$G^* = \frac{G_o}{(\omega + \frac{3}{2})}. \tag{4.105b}$$

Choosing a spherically symmetric metric element

$$ds^2 = -e^{2\bar{\nu}}dt^2 + e^{2\bar{\lambda}}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \tag{4.106}$$

permits one to compute the following expression for $\bar{T}_s^{tt}$, viz.

$$\bar{T}_s^{tt} = \frac{c^4}{16\pi G^* \phi^2}\left(e^{-4\bar{\nu}}\left(\frac{\partial\phi}{\partial t}\right)^2 + e^{-2(\bar{\nu}+\bar{\lambda})}\left(\frac{\partial\phi}{\partial r}\right)^2\right). \tag{4.107}$$

The scalar field may be reabsorbed into the metric coefficients yielding the result

$$\bar{T}_s^{tt} = \frac{c^4}{16\pi G^*}\left(e^{-4\nu}\left(\frac{\partial\phi}{\partial t}\right)^2 + e^{-2(\nu+\lambda)}\left(\frac{\partial\phi}{\partial r}\right)^2\right), \tag{4.108}$$

which, as can be demonstrated, is equal to the time-time component of the scalar field stress-energy tensor in the variable G representation. Consequently, one can conclude that

$$\left(\frac{dE}{dt}\right)_s = \left(\frac{d\bar{E}}{d\bar{t}}\right)_s. \tag{4.109}$$

This result implies that the physics of the system is invariant with respect to choice of representation.

From Brans and Dicke (1961) we have the contravariant metric components

$$g^{tt} = e^{-2\nu} = \frac{(1 + (M/2c^2\phi_o r)[(2\omega+4)/(2\omega+3)]^{1/2})^{2/\mu}}{(1 - (M/2c^2\phi_o r)[(2\omega+4)/(2\omega+3)]^{1/2})^{2/\mu}} \tag{4.110a}$$

and

$$g^{rr} = e^{-2\lambda} =$$

$$\frac{1}{(1 + (M/2c^2\phi_o r)[(2\omega+4)/(2\omega+3)]^{1/2})^4}$$
$$\times\frac{(1 + (M/2c^2\phi_o r)[(2\omega+4)/(2\omega+3)]^{1/2})^{2[(\mu-C-1)/\mu]}}{(1 - (M/2c^2\phi_o r)[(2\omega+4)/(2\omega+3)]^{1/2})^{2[(\mu-C-1)/\mu]}}, \tag{4.110b}$$

both of which are positive definite in the far-field limit. The constant expressions

$$\mu = [(C+1)^2 - C(1 - \frac{1}{2}\omega C)]^{1/2} \tag{4.111a}$$

and

$$C = -1/(2+\omega) \tag{4.111b}$$

are used in equations (4.110a) and (4.110b). Therefore, one may conclude that

$$\bar{T}_s^{tt} \geq 0, \tag{4.112}$$

which implies the following conditions:

$$\left(\frac{d\bar{E}}{d\bar{t}}\right)_s \leq 0 \quad \frac{d}{dt}\delta^{(2)}H \leq 0. \tag{4.113}$$

One may deduce the following theorem, namely: the negative energy extremal of a spherical matter configuration can manifest run-away energetic behavior with increasingly negative energies and a gradual slide away from equilibrium. However, this run-away instability hinges upon the fact that a negative energy extremal does indeed exist for a spherical matter configuration. An interesting point made by Wagoner (1970) and Nutku (1969) is that there is no scalar field energy loss at the order of the post-Newtonian approximation. Scalar energy is only lost at higher than the post-Newtonian approximation.

CHAPTER 5
THE VLASOV-KIBBLE SYSTEM

In 1956, Utiyama (1956) tried to show that General Relativity could be formulated as a Yang-Mills theory (Yang and Mills (1954)) of the homogeneous Lorentz group. Corresponding to the six space-time dependent parameters $\epsilon^{ij}(x) = -\epsilon^{ji}(x)$ of the homogeneous Lorentz group are six vector gauge fields $A^{ij}{}_{\mu}$. Utiyama, on an *a priori* basis, introduced curvilinear coordinates and a tetrad of vectors $h_k{}^{\mu}$ corresponding to a *vierbein* system of gauge fields. Therefore, the attempt to show that General Relativity is a Yang-Mills theory was only partially successful because no explanation for the *a priori* introduction of the curvilinear coordinate system and *vierbein* tetrad $h_k{}^{\mu}$ was given. Furthermore, it was not possible to relate the Yang-Mills gauge fields $A^{ij}{}_{\mu}$ to the Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu}$ in a unique manner except by making an *ad hoc* assumption that the quantities $\Gamma'^{\lambda}{}_{\mu\nu}$ calculated from $A^{ij}{}_{\mu}$ are symmetric.

In 1961, Kibble (1961) demonstrated that the *vierbein* fields $h_k{}^{\mu}$ could naturally appear in a gravitational Yang-Mills theory through an extension of the gauge group to the inhomogeneous 10-parameter Lorentz (Poincaré) group. The tetrad of gauge fields $h_k{}^{\mu}$ would then be associated with the four additional translation parameters ϵ^k of the (extended) Lorentz group. Kibble was able to prove that a gravitational theory, which is remarkably similar to General Relativity, can be derived as a Yang-Mills theory of the Poincaré group associated with a Minkowskian space-time. We will attempt, here, to provide a Hamiltonian formulation of the Vlasov-Kibble gauge invariant gravity system. In Kandrup and O'Neill (1994), a Hamiltonian function was utilized in which the gravitational field is treated on the same

footing as the Boltzmann distribution f. For this theory, an analogous formulation will be employed in which a *vierbein* $b^k{}_\mu$ (inverse :$h_k{}^\mu, b^k{}_\mu h_k{}^\nu = \delta_\mu{}^\nu$) and a local affine connection $A^{ij}{}_\mu$ ($A^{ij}{}_\mu = -A^{ji}{}_\mu$) function as dynamical variables.

In this chapter, then, there will be five different dynamical variables, namely: (1) the tetrad field $b^k{}_\mu$ and its (2) conjugate momentum $\Pi_k{}^\mu$, as well as (3) the local affine connection $A^{ij}{}_\mu$ and its (4) conjugate momentum $\Pi_{ij}{}^\mu$, and (5) the Boltzmann distribution function f. As in Kandrup and O'Neill (1994) one may identify a Hamiltonian function H and an extended bracket $\langle A, B\rangle$. These are chosen so that the equations of constraint $\partial_t F = \langle F, H\rangle$, for all functionals $F = F[b^k{}_\mu, \Pi_k{}^\mu; A^{ij}{}_\mu, \Pi_{ij}{}^\mu; f]$, are equivalent to the correct dynamical equations of motion for $b^k{}_\mu, \Pi_k{}^\mu, A^{ij}{}_\mu, \Pi_{ij}{}^\mu$ and f.

5.1 Kibble Gauge Invariant Theory of Gravity

One may consider an arbitrary matter field Lagrangian which is invariant with respect to the global Poincaré group

$$\mathcal{L}^M = \mathcal{L}^M(\chi, \partial_k\chi), \tag{5.1}$$

with $\partial_k\chi = \partial\chi/\partial x^k$ and χ a column vector that is transformation invariant with respect to some representation of the Lorentz group. This theory will remain invariant with respect to the local Poincaré group if we introduce a tetrad field $b^k{}_\mu$ and a local affine connection $A^{ij}{}_\mu$ which corresponds to the six Lorentz parameters ϵ^{ij} ($\epsilon^{ij} = -\epsilon^{ji}$) of the same group. The covariant derivative of the matter field has the following form (cf. Kibble (1961))

$$D_k\chi = h_k{}^\mu \nabla_\mu\chi = h_k{}^\mu(\partial_\mu + \frac{1}{2}A^{ij}{}_\mu S_{ij})\chi, \tag{5.2}$$

with $h_k{}^\mu$ denoting the inverse tetrad field ($b^k{}_\mu h_k{}^\nu = \delta_\mu{}^\nu$) and S_{ij} denoting the Lorentz group generators. The gauge invariant matter Lagrangian will be

$$\mathcal{L}^m = b\mathcal{L}^M(\chi, D_k\chi) = b\mathcal{L}^M(\chi, \chi_{;k}), \tag{5.3}$$

where $b = \det b^k{}_\mu$ and $D_k\chi = \chi_{;k}$. From the two gauge fields one may construct two different types of gauge field strengths, viz. (1) the torsion

$$T^i{}_{\mu\nu} = 2\nabla_{[\nu}b^i{}_{\mu]} = 2(b^i{}_{[\mu,\nu]} + A^i{}_{l[\nu}b^l{}_{\mu]}) \tag{5.4}$$

and (2) the curvature

$$R^{ij}{}_{\mu\nu} = 2(A^{ij}{}_{[\mu,\nu]} + A^i{}_{n[\nu}A^{nj}{}_{\mu]}). \tag{5.5}$$

The notation $X_{[\mu,\nu]}$ denotes the condition of antisymmetry, viz.

$$X_{[\mu,\nu]} = \frac{1}{2}(X_{\mu,\nu} - X_{\nu,\mu}). \tag{5.6}$$

One may take the gravitational Lagrangian density to possess the general form (Nikolić (1984))

$$\mathcal{L}^g = b\mathcal{L}^G(T_{ijk}, R_{ijkl}). \tag{5.7}$$

In this expression we have converted Greek indices to Latin indices by contraction with the inverse tetrad field $h_k{}^\mu$, e.g., $T^k{}_{m\nu} = h_m{}^\mu T^k{}_{\mu\nu}$. Latin indices correspond to local Lorentz (anholonomic) indices, while Greek indices correspond to coordinate indices (holonomic). The first letters of these alphabets (a, b, c, ...; α, β, γ,...) run over values 1, 2, 3, but other letters take on the values 0, 1, 2, 3. The diagonal of the Minkowski metric η_{ij} corresponds to (+, -, -, -). The antisymmetric tensors ϵ^{ijkl} and ϵ^{abc} possess the values $\epsilon^{0123} = \epsilon^{123} = 1$. The Ricci tensor can be defined as $R_{ij} = R^k{}_{ikj}$. Making the rather typical assumption of a matter field that minimally couples to gravity leads to a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}^m + \mathcal{L}^g. \tag{5.8}$$

From Kibble (1961), one may define the currents

$$\mathcal{T}^k{}_\mu = \partial\mathcal{L}^m/\partial h_k{}^\mu = bb^i{}_\mu\{(\partial\mathcal{L}^M/\partial\chi_{;k})\chi_{;i} - \delta^k{}_i\mathcal{L}^M\}, \tag{5.9a}$$

$$\Sigma^{\mu}{}_{ij} = -2(\partial\mathcal{L}^m/\partial A^{ij}{}_{\mu}) = -bh_k{}^{\mu}(\partial\mathcal{L}^M/\partial\chi_{;k})S_{ij}\chi, \tag{5.9b}$$

where, from eqs. (5.2) and (5.3), we have

$$\mathcal{L}^m = \mathcal{L}^m\{\chi, \chi_{,\mu}, h_k{}^{\mu}, A^{ij}{}_{\mu}\} = b\mathcal{L}^M\{\chi, \chi_{;k}\}. \tag{5.10}$$

For a gravitational Lagrangian of the form

$$\mathcal{L}^g = bR^{ij}{}_{ij} = bR, \tag{5.11}$$

the conservation laws (Kibble (1961)) possess the form

$$(\mathcal{T}^k{}_{\nu}h_k{}^{\mu})_{|\mu} + \mathcal{T}^k{}_{\mu}h_k{}^{\mu}{}_{|\nu} = \frac{1}{2}\Sigma^{\mu}{}_{ij}R^{ij}{}_{\mu\nu}, \tag{5.12a}$$

$$\Sigma^{\mu}{}_{ij|\mu} = \mathcal{T}_{i\mu}h_j{}^{\mu} - \mathcal{T}_{j\mu}h_i{}^{\mu}. \tag{5.12b}$$

From Kibble (1961), the equations of motion are

$$b(R^{ik}{}_{jk} - \frac{1}{2}\delta^i{}_jR) = -\mathcal{T}^i{}_{\mu}h_j{}^{\mu}, \tag{5.13a}$$

$$-[b(h_i{}^{\mu}h_j{}^{\nu} - h_j{}^{\mu}h_i{}^{\nu})]_{|\nu} = \Sigma^{\mu}{}_{ij}, \tag{5.13b}$$

where

$$R^{ij}{}_{kl} = h_k{}^{\mu}h_l{}^{\nu}R^{ij}{}_{\mu\nu}. \tag{5.14}$$

5.2 Vlasov-Kibble Hamiltonian Structure and Formulation

The dynamical variables that we deal with primarily are the tetrad field $b^k{}_{\mu}$ and affine connection $A^{ij}{}_{\mu}$, along with their respective momenta $\Pi_k{}^{\mu}$ and $\Pi_{ij}{}^{\mu}$. Torsion and curvature are defined in terms of antisymmetric derivatives of $b^k{}_{\mu}$ and $A^{ij}{}_{\mu}$. Consequently, the velocities $b^k{}_{0,0}$ and $A^{ij}{}_{0,0}$ do not appear and we have the corresponding primary constraints, viz.

$$\Pi_k{}^0 = 0, \tag{5.15a}$$

$$\Pi_{ij}{}^0 = 0. \tag{5.15b}$$

First of all, one must consider how to pass from the inverse tetrad field $h_k{}^\mu$ to the following set of variables $n_k, h_{\bar{k}}{}^\alpha, N, N^\alpha$. With respect to a local Lorentz basis, the components of the unit normal vector $\vec{n}$ to the $x^0 =$ const. hypersurface are given by

$$n_k = h_k{}^0/\sqrt{g^{00}}, \tag{5.16}$$

where

$$g^{\mu\nu} = h_k{}^\mu h^{k\nu}. \tag{5.17}$$

Following Nikolić (1984) it is possible to decompose the inverse tetrad $h_k{}^\mu$ into components orthogonal and parallel with respect to the local Lorentz indices:

$$h_k{}^\mu = h_{\bar{k}}{}^\mu + n_k h_\perp{}^\mu, \tag{5.18a}$$

$$h_{\bar{k}}{}^\mu \equiv \delta_{\bar{k}}{}^l h_l{}^\mu = (\delta_k{}^l - n_k n^l) h_l{}^\mu, \tag{5.18b}$$

$$h_\perp{}^\mu \equiv n^k h_k{}^\mu = n^\mu. \tag{5.18c}$$

One may see that

$$h_{\bar{k}}{}^0 \equiv 0, \quad h_{\bar{k}}{}^\alpha = {}^{(3)}g^{\alpha\beta} b_{k\beta}, \tag{5.19}$$

where ${}^{(3)}g^{\alpha\beta}$ corresponds to the three-dimensional contravariant metric. We may infer that n_k and $h_{\bar{k}}{}^\alpha$ do not depend on $b^k{}_0$. Introduction of the lapse

$$N \equiv 1/\sqrt{g^{00}} = n_k b^k{}_0 \tag{5.20}$$

and shift functions

$$N^\alpha \equiv -g^{0\alpha}/g^{00} = h_{\bar{k}}{}^\alpha b^k{}_0 \tag{5.21}$$

allows one to write equation (5.18c) as

$$h_\perp{}^0 = 1/N, \quad h_\perp{}^\alpha = -N^\alpha/N. \tag{5.22}$$

Therefore, we can pass from $h_k{}^\mu$ to $\{n_k, h_{\bar{k}}{}^\alpha, N, N^\alpha\}$ without difficulty. A similar decomposition of $b^k{}_\mu$ leads to the following results

$$t^\mu b^k{}_\mu = b^k{}_0 = b^{\bar{k}}{}_0 + n^k b^\perp{}_0 = N^k + n^k N \tag{5.23a}$$

and

$$b^k{}_\alpha = b^{\bar{k}}{}_\alpha + n^k b^\perp{}_\alpha = b^{\bar{k}}{}_\alpha. \tag{5.23b}$$

The purely gravitational part of the canonical Hamiltonian can be written as (Nikolić (1984))

$$\mathcal{H}^G{}_{can} = \Pi_k{}^\alpha b^k{}_{\alpha,0} + \frac{1}{2}\Pi_{ij}{}^\alpha A^{ij}{}_{\alpha,0} - b\mathcal{L}^G. \tag{5.24}$$

It is a straightforward exercise to derive the following velocities (Nikolić (1984))

$$b^k{}_{\alpha,0} = Nb^l{}_\alpha T^k{}_{\bar{l}\perp} + N^\beta T^k{}_{\alpha\beta} - T^k{}_{\alpha 0}(0), \tag{5.25a}$$

$$A^{ij}{}_{\alpha,0} = Nb^l{}_\alpha R^{ij}{}_{\bar{l}\perp} + N^\beta R^{ij}{}_{\alpha\beta} - R^{ij}{}_{\alpha 0}(0), \tag{5.25b}$$

where

$$T^k{}_{\alpha 0}(0) = -b^k{}_{0,\alpha} + b^i{}_\alpha A^k{}_{i0} - b^i{}_0 A^k{}_{i\alpha}, \tag{5.26a}$$

$$R^{ij}{}_{\alpha 0}(0) = -A^{ij}{}_{0,\alpha} + A^i{}_{l0} A^{lj}{}_\alpha - A^i{}_{l\alpha} A^{lj}{}_0. \tag{5.26b}$$

These velocities contain spatial derivatives of the unphysical variables $b^k{}_0$ and $A^{ij}{}_0$ (these unphysical variables correspond to the momentum constraints (5.15a, b)). A simple calculation (Nikolić (1984)) yields the following canonical Hamiltonian

$$\mathcal{H}^G{}_{can} = N\mathcal{H}^G{}_\perp + N^\alpha \mathcal{H}^G{}_\alpha - \frac{1}{2}A^{ij}{}_0 \mathcal{H}^G{}_{ij} + D^{G\alpha}{}_{,\alpha}, \tag{5.27}$$

where

$$D^{G\alpha}{}_{,\alpha} = \left(b^k{}_0 \Pi_k{}^\alpha + \frac{1}{2}A^{ij}{}_0 \Pi_{ij}{}^\alpha\right)_{,\alpha}, \tag{5.28a}$$

$$\mathcal{H}^G{}_{ij} = 2\Pi_{[i}{}^{\alpha} b_{j]\alpha} + \nabla_\alpha \Pi_{ij}{}^{\alpha}, \tag{5.28b}$$

$$\mathcal{H}^G{}_{\alpha} = \Pi_k{}^{\beta} T^k{}_{\beta\alpha} - b^k{}_{\alpha} \nabla_\beta \Pi_k{}^{\beta} + \frac{1}{2} \Pi_{ij}{}^{\beta} R^{ij}{}_{\beta\alpha}, \tag{5.28c}$$

$$\mathcal{H}^G{}_{\perp} = J\left[\frac{1}{J}\Pi_k{}^{\bar{l}} T^k{}_{\bar{l}\perp} + \frac{1}{2J}\Pi_{ij}{}^{\bar{l}} R^{ij}{}_{\bar{l}\perp} - \bar{\mathcal{L}}^G\right] - n^k \nabla_\alpha \Pi_k{}^{\alpha}. \tag{5.28d}$$

In equation (5.28d) the notations $\Pi_k{}^{\bar{l}} = \Pi_k{}^{\alpha} b^l{}_{\alpha}$ and $\Pi_{ij}{}^{\bar{l}} = \Pi_{ij}{}^{\alpha} b^l{}_{\alpha}$ have been employed. The unphysical variables make their appearance in the divergence term (5.28a). The bracketed expression in equation (5.28d) can be written as a function of $T^k{}_{\bar{l}\bar{m}}$, $\Pi_k{}^{\bar{l}}/J$, $R^{ij}{}_{\bar{k}\bar{l}}$, $\Pi_{ij}{}^{\bar{k}}/J$ and n^k , i.e., velocities $T^k{}_{\bar{l}\perp}$ and $R^{ij}{}_{\bar{k}\perp}$ can be eliminated from it. This will be shown later. Therefore, it is independent of unphysical variables and corresponds to the only dynamical part of the gravitational canonical Hamiltonian.

Using equations (5.20) and (5.21) we may rewrite the canonical Hamiltonian in the form

$$\begin{aligned}
\mathcal{H}^G{}_{can} &= n_k b^k{}_0 \mathcal{H}^G{}_{\perp} + h_{\bar{k}}{}^{\alpha} b^k{}_0 \mathcal{H}^G{}_{\alpha} - \frac{1}{2} A^{ij}{}_0 \mathcal{H}^G{}_{ij} + D^{G\alpha}{}_{,\alpha} \\
&= b^k{}_0 (n_k \mathcal{H}^G{}_{\perp} + h_{\bar{k}}{}^{\alpha} \mathcal{H}^G{}_{\alpha}) - \frac{1}{2} A^{ij}{}_0 \mathcal{H}^G{}_{ij} + D^{G\alpha}{}_{,\alpha} \\
&= b^k{}_0 \mathcal{H}^G{}_{k} - \frac{1}{2} A^{ij}{}_0 \mathcal{H}^G{}_{ij} + D^{G\alpha}{}_{,\alpha},
\end{aligned} \tag{5.29}$$

so, apart from the divergence term, $\mathcal{H}^G{}_{can}$ has linear dependence on the unphysical variables $b^k{}_0$ and $A^{ij}{}_0$. One can revert to the ADM form by using the transformations

$$\mathcal{H}_{\perp} = n^k \mathcal{H}_k, \tag{5.30a}$$

$$\mathcal{H}_{\alpha} = b^k{}_{\alpha} \mathcal{H}_k. \tag{5.30b}$$

One must, also, examine the role which the primary constraints (5.15a, b) play in the Hamiltonian structure of the theory. The total gravitational Hamiltonian may be initially written as

$$\mathcal{H}_{tot} = \Pi_k{}^{\mu} b^k{}_{\mu,0} + \frac{1}{2} \Pi_{ij}{}^{\mu} A^{ij}{}_{\mu,0} - b\mathcal{L}^G. \tag{5.31}$$

Since $b^k{}_0$ and $A^{ij}{}_0$ are unphysical variables, this expression may be rewritten as

$$\mathcal{H}_{tot} = \Pi_k{}^\alpha b^k{}_{\alpha,0} + \frac{1}{2}\Pi_{ij}{}^\alpha A^{ij}{}_{\alpha,0} - b\mathcal{L}^G + \Pi_k{}^0 b^k{}_{0,0} + \frac{1}{2}\Pi_{ij}{}^0 A^{ij}{}_{0,0}$$

$$= N\mathcal{H}_\perp + N^\alpha \mathcal{H}_\alpha - \frac{1}{2}A^{ij}{}_0\mathcal{H}_{ij} + D^\alpha{}_{,\alpha} + \Pi_k{}^0 b^k{}_{0,0} + \frac{1}{2}\Pi_{ij}{}^0 A^{ij}{}_{0,0}, \tag{5.32}$$

where $\Pi_k{}^0$ and $\Pi_{ij}{}^0$ are the primary constraints (5.15a, b). Following Nikolić (1984) we will consider $b^k{}_{0,0}$ and $A^{ij}{}_{0,0}$ to be arbitrary multipliers, denoted respectively as $u^k{}_0$ and $u^{ij}{}_0$.

Starting with the basic dynamical variables $b^k{}_\alpha$, $\Pi_k{}^\alpha$, $A^{ij}{}_\alpha$, $\Pi_{ij}{}^\alpha$ and f, one can define the cosymplectic structure as a bracket $\langle F, G\rangle$ which acts on functional pairs such as $F[b^k{}_\alpha, \Pi_k{}^\alpha; A^{ij}{}_\alpha, \Pi_{ij}{}^\alpha; f]$ and $G[b^k{}_\alpha, \Pi_k{}^\alpha; A^{ij}{}_\alpha, \Pi_{ij}{}^\alpha; f]$. One can postulate a bracket of the form

$$\begin{aligned}\langle F, G\rangle[b^k{}_\alpha, \Pi_k{}^\alpha; A^{ij}{}_\alpha, \Pi_{ij}{}^\alpha; f] = 16\pi \int d^3x \Big(\frac{\delta F}{\delta b^k{}_\alpha}\frac{\delta G}{\delta \Pi_k{}^\alpha} - \frac{\delta F}{\delta \Pi_k{}^\alpha}\frac{\delta G}{\delta b^k{}_\alpha}\Big) \\ +16\pi \int d^3x \Big(\frac{\delta F}{\delta A^{ij}{}_\alpha}\frac{\delta G}{\delta \Pi_{ij}{}^\alpha} - \frac{\delta F}{\delta \Pi_{ij}{}^\alpha}\frac{\delta G}{\delta A^{ij}{}_\alpha}\Big) \\ -16\pi \int d^3x \Big(\frac{\delta F}{\delta A^{ji}{}_\alpha}\frac{\delta G}{\delta \Pi_{ij}{}^\alpha} - \frac{\delta F}{\delta \Pi_{ij}{}^\alpha}\frac{\delta G}{\delta A^{ji}{}_\alpha}\Big) \\ + \int d\Gamma f\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\},\end{aligned} \tag{5.33}$$

where $\delta/\delta X$ denotes the usual functional derivative with respect to the variables X and

$$\{A, B\} = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial x^a}\frac{\partial A}{\partial p_a} \tag{5.34}$$

is the ordinary canonical Poisson bracket. The form of equation (5.33) is chosen to satisfy the Poisson bracket (Nikolić (1984))

$$\{A^{ij}{}_\alpha, \Pi_{kl}{}^\beta\} = 2\delta^i{}_{[k}\delta^j{}_{l]}\delta^\beta{}_\alpha \delta(\vec{x} - \vec{x}'). \tag{5.35}$$

It is straightforward to demonstrate that equation (5.33) constitutes a *bona fide* Lie bracket because it is antisymmetric and satisfies the Jacobi Identity

$$\langle F, \langle G, H\rangle\rangle + \langle G, \langle H, F\rangle\rangle + \langle H, \langle F, G\rangle\rangle = 0. \tag{5.36}$$

The requirement that the equations of constraint

$$\partial_t F = \langle F, H \rangle, \tag{5.37}$$

which hold for arbitrary functionals F, generate the correct dynamical equations implies that the Lie bracket and Hamiltonian combine to generate the Vlasov-Kibble gauge invariant gravity system. Through simple construction, we derive the expressions

$$\partial_t b^k{}_\alpha = \langle b^k{}_\alpha, H \rangle = 16\pi \frac{\delta H}{\delta \Pi_k{}^\alpha}, \tag{5.38a}$$

$$\partial_t \Pi_k{}^\alpha = \langle \Pi_k{}^\alpha, H \rangle = -16\pi \frac{\delta H}{\delta b^k{}_\alpha}, \tag{5.38b}$$

$$\partial_t A^{ij}{}_\alpha = \langle A^{ij}{}_\alpha, H \rangle = 32\pi \frac{\delta H}{\delta \Pi_{ij}{}^\alpha} \tag{5.38c}$$

and

$$\partial_t \Pi_{ij}{}^\alpha = \langle \Pi_{ij}{}^\alpha, H \rangle = -32\pi \frac{\delta H}{\delta A^{ij}{}_\alpha}. \tag{5.38d}$$

Explicit forms for expressions (5.38a) to (5.38d) may be derived. Before doing this, however, one must analyze the explicit form of the matter Hamiltonian. Following Nikolić (1984) we will assume minimal coupling between the matter and gauge fields so that the Hamiltonian is a sum of matter and gravitational parts. The covariant derivative $D_k\chi$ contains the time derivative $\chi_{,0}$ (this covariant derivative is present in the gauge invariant matter field Lagrangian (5.3)). One may decompose $D_k\chi$ into orthogonal and parallel components

$$D_k\chi = n_k D_\perp \chi + D_{\bar{k}}\chi = n_k h_\perp{}^\mu \nabla_\mu \chi + h_{\bar{k}}{}^\alpha \nabla_\alpha \chi. \tag{5.39}$$

The covariant derivative $D_{\bar{k}}\chi$ does not depend on velocities or unphysical variables. The matter field Lagrangian (5.3) may be reexpressed in terms of these new variables as

$$\mathcal{L}^M = \bar{\mathcal{L}}^M(\chi, D_\perp\chi, D_{\bar{k}}\chi; n^k). \tag{5.40}$$

All the dependence of equation (5.40) on velocities and unphysical variables is through $D_\perp\chi$. It is a straightforward exercise to derive the following velocity of the matter field (Nikolić (1984))

$$\chi_{,0} = ND_\perp\chi + N^\alpha\nabla_\alpha\chi - \frac{1}{2}A^{ij}{}_0S_{ij}\chi. \tag{5.41}$$

The determinant b may be factorized as

$$b = \det b^k{}_\mu = \frac{N}{n_0}\det b^a{}_\alpha = NJ \tag{5.42}$$

which will enable us to calculate the corresponding field momentum

$$\Pi \equiv \frac{\partial(b\mathcal{L}^M)}{\partial\chi_{,0}} = J\frac{\partial\bar{\mathcal{L}}^M}{\partial D_\perp\chi}. \tag{5.43}$$

The canonical matter Hamiltonian has the form

$$\mathcal{H}^M{}_{can} = \Pi\chi_{,0} - b\mathcal{L}^M, \tag{5.44}$$

which, upon substitution of equation (5.41), yields the following ADM components

$$\mathcal{H}^M{}_{ij} = \Pi S_{ij}\chi, \tag{5.45a}$$

$$\mathcal{H}^M{}_\alpha = \Pi\nabla_\alpha\chi, \tag{5.45b}$$

$$\mathcal{H}^M{}_\perp = \Pi D_\perp\chi - J\bar{\mathcal{L}}^M = J\Big[\frac{\partial\bar{\mathcal{L}}^M}{\partial D_\perp\chi}D_\perp\chi - \bar{\mathcal{L}}^M\Big], \tag{5.45c}$$

and

$$D^{M\alpha}{}_{,\alpha} \equiv 0. \tag{5.45d}$$

It should be noted that $\mathcal{H}^M{}_\perp$ is the Legendre transformation of $\bar{\mathcal{L}}^M$ with respect to $D_\perp\chi$. Thus, it can be expressed as a function of χ, $D_{\bar{k}}\chi$, n^k and Π/J. Since we are employing a statistical mechanics formulation (involving a Boltzmann distribution function) to describe

the matter Hamiltonian, we are immediately presented with a problem. If we use the matter Hamiltonian (cf. Kandrup and O'Neill (1994))

$$H_M = \int d\Gamma f \varepsilon, \tag{5.46}$$

where $\varepsilon = -p_t$, the expression for ε is typically derived from the mass shell constraint

$$g^{\mu\nu} p_\mu p_\nu = -m^2, \tag{5.47}$$

which, in our *vierbein* formulation, can be written as

$$h_k{}^\mu h^{k\nu} p_\mu p_\nu = -m^2. \tag{5.48}$$

This expression may be subjected to the standard ADM decomposition

$$H_M = \int d^3x \{N\rho + N^\alpha J_\alpha\}, \tag{5.49}$$

where

$$\mathcal{H}^M{}_\perp = \rho, \tag{5.50a}$$

$$\mathcal{H}^M{}_\alpha = J_\alpha. \tag{5.50b}$$

Clearly, it seems that we are lacking an extra term $\mathcal{H}^M{}_{ij}$ in the ADM decomposition; with this term the ADM decomposition would possess the form

$$H_M = \int d^3x \{N\rho + N^\alpha J_\alpha - \frac{1}{2} A^{ij}{}_0 \mathcal{H}^M{}_{ij}\}. \tag{5.51}$$

To make an appropriate modification of the matter Hamiltonian, we may consider the representation of a perfect fluid in terms of a set of six potentials (Schutz (1971))

$$P^\mu = mu^\mu = g^{\mu\nu} \frac{m}{\omega} (\phi_{,\nu} + \alpha \beta_{,\nu} + \theta S_{,\nu}). \tag{5.52}$$

Using the covariant derivative prescription (5.2), one may modify this expression for the contravariant four-momentum to

$$P^\mu = g^{\mu\nu} \frac{m}{\omega} (\phi_{,\nu} + \alpha \beta_{,\nu} + \theta S_{,\nu}) + \frac{g^{\mu\nu}}{2} A^{ij}{}_\nu \frac{m}{\omega'} (S_{ij} \phi + \alpha S_{ij} \beta + \theta S_{ij} S). \tag{5.53}$$

The Lorentz group generators S_{ij} may be formulated in a coordinate representation as

$$S_{ij} = x_i \partial_j - x_j \partial_i. \tag{5.54}$$

Therefore, the expression (5.53) may be written as

$$P^\mu = g^{\mu\nu}(p_\nu + \frac{1}{2}A^{ij}{}_\nu L_{ij}), \tag{5.55}$$

where L_{ij} corresponds to the orbital angular momentum.

This leads to a mass shell constraint

$$g^{\mu\nu}(p_\mu + \frac{1}{2}A^{ij}{}_\mu L_{ij})(p_\nu + \frac{1}{2}A^{kl}{}_\nu L_{kl}) = -m^2. \tag{5.56}$$

This constraint is an analog to the mass shell constraint of electromagnetism

$$g^{\mu\nu}(p_\mu + eA_\mu)(p_\nu + eA_\nu) = -m^2. \tag{5.57}$$

In our *vierbein* formulation, the mass shell constraint may be written as

$$h^{k\mu}h_k{}^\nu(p_\mu + \frac{1}{2}A^{ij}{}_\mu L_{ij})(p_\nu + \frac{1}{2}A^{kl}{}_\nu L_{kl}) = -m^2. \tag{5.58}$$

The analogy with electromagnetism is of importance. Electromagnetism can be viewed (theoretically) as a gauge theory of the group $U(1)$. However, there is no *a priori* reason for the prescription

$$p_\mu \to p_\mu + eA_\mu$$

in the context of classical physics. The prescription is due to gauge invariance in quantum mechanics. Likewise, there is no basis in classical physics for the prescription

$$p_\mu \to p_\mu + \frac{1}{2}A^{ij}{}_\mu L_{ij}.$$

The derivation of this prescription from the perfect fluid representation should be seen only as a heuristic derivation. The charges L_{ij} may or may not correspond to orbital angular momentum. Only experimentation can detect them and prove their existence.

It follows that the energy of a single massive particle will be of the form

$$E = -p_t - \frac{1}{2} A^{ij}{}_0 L_{ij} = \varepsilon - \frac{1}{2} A^{ij}{}_0 L_{ij}. \tag{5.59}$$

Therefore, we may write the matter Hamiltonian H_M (5.46) as

$$H_M = \int d\Gamma f E = \int d\Gamma f \varepsilon - \frac{1}{2} \int d\Gamma f A^{ij}{}_0 L_{ij}. \tag{5.60}$$

This yields the desired form for H_M (5.46), viz.

$$\begin{aligned} H_M &= \int d^3x d^3p dm f \varepsilon - \frac{1}{2} \int d^3x A^{ij}{}_0 \int d^3p dm f L_{ij} \\ &= \int d^3x (N\rho + N^\alpha J_\alpha - \frac{1}{2} A^{ij}{}_0 \mathcal{H}^M{}_{ij}). \end{aligned} \tag{5.61}$$

One may derive explicit expressions for the dynamical equations (5.38). For the physical variables $b^l{}_\gamma$ and $A^{qr}{}_\gamma$ we have

$$\partial_t b^l{}_\gamma = N b^m{}_\gamma T^l{}_{\bar{m}\perp} + N^\alpha T^l{}_{\gamma\alpha} + b^l{}_{0,\gamma} + b^k{}_0 A^l{}_{k\gamma} - A^{lj}{}_0 b_{j\gamma} \tag{5.62a}$$

and

$$\begin{aligned} \partial_t A^{qr}{}_\gamma &= N R^{qr}{}_{\bar{m}\perp} b^m{}_\gamma + N^\alpha R^{qr}{}_{\gamma\alpha} \\ &+ A^{qr}{}_{0,\gamma} - A^q{}_{j0} A^{jr}{}_\gamma + A^{ir}{}_0 A^q{}_{i\gamma}. \end{aligned} \tag{5.62b}$$

These expressions were previously derived. It is important to note the link between these dynamical equations and their conjugate momenta. The momentum expressions for $b^k{}_\mu$ and $A^{ij}{}_\mu$ are

$$\Pi_k{}^{\bar{l}} = b^l{}_\alpha \frac{\partial (b\mathcal{L}^G)}{\partial b^k{}_{\alpha,0}} = J \frac{\partial \bar{\mathcal{L}}^G}{\partial T^k{}_{\bar{l}\perp}} \tag{5.63a}$$

and

$$\Pi_{ij}{}^{\bar{k}} = b^k{}_\alpha \frac{\partial (b\mathcal{L}^G)}{\partial A^{ij}{}_{\alpha,0}} = J \frac{\partial \bar{\mathcal{L}}^G}{\partial R^{ij}{}_{\bar{k}\perp}}, \tag{5.63b}$$

where $\Pi_k{}^{\bar{l}} = \Pi_k{}^\alpha b^l{}_\alpha$ and $\Pi_{ij}{}^{\bar{k}} = \Pi_{ij}{}^\alpha b^k{}_\alpha$. The Lagrangian $\bar{\mathcal{L}}^G$ is the same as that in equation (5.7), except that the torsion and curvature have been decomposed as

$$T^k{}_{lm} = 2T^k{}_{[\bar{l}\perp}n_{m]} + T^k{}_{\bar{l}\bar{m}} \tag{5.64a}$$

and

$$R^{ij}{}_{kl} = 2R^{ij}{}_{[\bar{k}\perp}n_{l]} + R^{ij}{}_{\bar{k}\bar{l}}. \tag{5.64b}$$

The Lagrangian (5.11) has no torsion dependence and so $\Pi_k{}^\alpha \equiv 0$. It follows that $\partial_t b^l{}_\gamma \equiv 0$ by setting $\Pi_k{}^\alpha$ equal to zero in the ADM Hamiltonian decomposition (5.28). The corresponding equations for the conjugate momenta $\Pi_k{}^\beta$ and $\Pi_{ij}{}^\beta$ are

$$\partial_t \Pi_k{}^\alpha = 0 \tag{5.65a}$$

and

$$\partial_t \Pi_{ij}{}^\alpha = [b(h_{\bar{i}}{}^\beta h_{\bar{j}}{}^\alpha - h_{\bar{i}}{}^\alpha h_{\bar{j}}{}^\beta)]_{|\beta} + [\Pi_{ij}{}^\alpha N^\beta]_{|\beta} - [\Pi_{ij}{}^\beta N^\alpha]_{|\beta} - \Sigma^\alpha{}_{ij} + \Pi_{kj}{}^\alpha A^k{}_{i0} - \Pi_{ki}{}^\alpha A^k{}_{j0}. \tag{5.65b}$$

Without torsion the momentum variable $\Pi_k{}^\alpha$ vanishes identically. The dynamical equation for $\Pi_{ij}{}^\alpha$ can be derived from the field equation (5.13b) by using expression (5.63b). We calculate the Vlasov equation to be

$$\partial_t f_o = \langle f_o, H\rangle = \{E, f_o\}. \tag{5.66}$$

This will take the form

$$\partial_t f_o = -\frac{P^\alpha}{P^t}\frac{\partial f_o}{\partial x^\alpha} + \frac{\partial g^{\mu\nu}}{\partial x^\alpha}\frac{P_\mu P_\nu}{2P^t}\frac{\partial f_o}{\partial p_\alpha} + \frac{P^\beta}{P^t}\frac{1}{2}\frac{\partial A^{ij}{}_\beta}{\partial x^\alpha}L_{ij}\frac{\partial f_o}{\partial p_\alpha} + \frac{P^\alpha}{P^t}\frac{1}{2}A^{ij}{}_\alpha\{L_{ij}, f_o\}, \tag{5.67}$$

where $P_\mu = p_\mu + \frac{1}{2}A^{ij}{}_\mu L_{ij}$. This is, in fact, the Vlasov equation in the gauge $A^{ij}{}_0 \equiv 0$. One may write

$$\{E, f_o\} = \{\varepsilon, f_o\} - \{\frac{1}{2}A^{ij}{}_0 L_{ij}, f_o\}$$

$$= \{\varepsilon, f_o\} - \frac{1}{2}\frac{\partial A^{ij}{}_0}{\partial x^\alpha} L_{ij} \frac{\partial f_o}{\partial p_\alpha} - \frac{1}{2} A^{ij}{}_0 \{L_{ij}, f_o\} \tag{5.68}$$

and obtain the full Vlasov equation

$$\partial_t f_o = \{\varepsilon, f_o\} = \frac{1}{2P^t}\frac{\partial g^{\mu\nu}}{\partial x^\alpha} P_\mu P_\nu \frac{\partial f_o}{\partial p_\alpha} + \frac{1}{2}\frac{P^\mu}{P^t}\frac{\partial A^{ij}{}_\mu}{\partial x^\alpha} L_{ij} \frac{\partial f_o}{\partial p_\alpha}$$

$$+ \frac{1}{2}\frac{P^\mu}{P^t} A^{ij}{}_\mu \{L_{ij}, f_o\} - \frac{P^\alpha}{P^t}\frac{\partial f_o}{\partial x^\alpha}. \tag{5.69}$$

We now wish to derive the Hamiltonian initial constraint equations. A variation of the unphysical variables $b^k{}_0$ and $A^{ij}{}_0$ in equation (5.29) yields

$$n_k \mathcal{H}_\perp + h_{\bar{k}}{}^\alpha \mathcal{H}_\alpha = -\int d^4p \frac{f}{m} P_k P^t \tag{5.70a}$$

and

$$2\Pi_{[i}{}^\alpha b_{j]\alpha} + \nabla_\alpha \Pi_{ij}{}^\alpha = 0. \tag{5.70b}$$

To derive these constraints, the following expressions have been utilized

$$\frac{\partial E}{\partial b^k{}_0} = -P_k \tag{5.71a}$$

and

$$\frac{\partial E}{\partial A^{ij}{}_0} = 0. \tag{5.71b}$$

Alternatively, one may derive the Hamiltonian initial constraint equations by variation of the lapse and shift variables $\{N, N^k\}$ present in equation (5.23a). The *vierbein* $b^k{}_0$ has four independent components and these correspond to the lapse N and three (independent) components of the shift vector N^k (remember that $n_k N^k = 0$). The constraints are

$$J\left[\frac{1}{J}\Pi_k{}^{\bar{l}} T^k{}_{\bar{l}\perp} + \frac{1}{2J}\Pi_{ij}{}^{\bar{l}} R^{ij}{}_{\bar{l}\perp} - \bar{\mathcal{L}}^G\right] - n^k \nabla_\alpha \Pi_k{}^\alpha$$

$$= 16\pi J \int \frac{d^3p dp_t}{NJ}\left[\frac{\left(E + N^\alpha(p_\alpha + \frac{1}{2}A^{kl}{}_\alpha L_{kl})\right)^2}{N^2}\right]\frac{f}{m}, \tag{5.72a}$$

$$h_{\bar{l}}{}^\alpha \{\Pi_k{}^\beta T^k{}_{\beta\alpha} - b^k{}_\alpha \nabla_\beta \Pi_k{}^\beta + \frac{1}{2}\Pi_{ij}{}^\beta R^{ij}{}_{\beta\alpha}\}$$

$$= -h_{\bar{l}}{}^{\alpha} 16\pi J \int \frac{d^3p dp_t}{NJ} \left[\frac{(p_\alpha + \frac{1}{2}A^{ij}{}_\alpha L_{ij})(E + N^\beta(p_\beta + \frac{1}{2}A^{kl}{}_\beta L_{kl}))}{N} \right] \frac{f}{m} \tag{5.72b}$$

and

$$2\Pi_{[i}{}^{\alpha} b_{j]\alpha} + \nabla_\alpha \Pi_{ij}{}^{\alpha} = 0, \tag{5.72c}$$

where the expressions

$$\frac{\partial E}{\partial N} = \frac{\left(E + N^\alpha(p_\alpha + \frac{1}{2}A^{kl}{}_\alpha L_{kl})\right)^2}{N^3 g^{t\mu}(p_\mu + \frac{1}{2}A^{kl}{}_\mu L_{kl})}, \tag{5.73a}$$

$$\frac{\partial E}{\partial N^k} = -P_{\bar{k}} = -h_{\bar{k}}{}^{\alpha} P_\alpha \tag{5.73b}$$

and

$$\frac{\partial E}{\partial A^{ij}{}_0} = 0 \tag{5.73c}$$

have been used in the above derivations. In Kibble gauge invariant gravity, the stress-energy tensor $T_{\mu\nu}$ is generally nonsymmetric because $h_k{}^{\mu}$ does not always appear in the matter Hamiltonian density $\mathcal{H}_M$ through the symmetric combination $g^{\mu\nu}$ (Kibble (1961)).

5.3 Initial Variation of the Hamiltonian

We may now consider the first variation of the Vlasov-Kibble Hamiltonian. This procedure will be analogous to the computations of chapters three and four. One may write the ADM Hamiltonian in the following format

$$H = H_G + H_M$$

$$= \int d^3x \{ NJ \left[\frac{1}{J} \Pi_k{}^{\bar{l}} T^k{}_{\bar{l}\perp} + \frac{1}{2J} \Pi_{ij}{}^{\bar{l}} R^{ij}{}_{\bar{l}\perp} - \bar{\mathcal{L}}^G \right]$$

$$-Nn^k \nabla_\alpha \Pi_k{}^{\alpha} + N^\alpha \Pi_k{}^{\beta} T^k{}_{\beta\alpha} - N^\alpha b^k{}_\alpha \nabla_\beta \Pi_k{}^{\beta} + \frac{N^\alpha}{2} \Pi_{ij}{}^{\beta} R^{ij}{}_{\beta\alpha}$$

$$-A^{ij}{}_0 \Pi_{[i}{}^{\alpha} b_{j]\alpha} - \frac{1}{2} A^{ij}{}_0 \nabla_\alpha \Pi_{ij}{}^{\alpha} \}$$

$$+ \int d^3x \left(b^k{}_0 \Pi_k{}^{\alpha} + \frac{1}{2} A^{ij}{}_0 \Pi_{ij}{}^{\alpha} \right)_{,\alpha} + \int d\Gamma f \varepsilon$$

$$-\frac{1}{2}\int d\Gamma A^{ij}{}_0 f L_{ij}. \tag{5.74}$$

The divergence term in equation (5.74) is readily converted into a surface integral

$$\oint d^2 s_\alpha (b^k{}_0 \Pi_k{}^\alpha + \frac{1}{2} A^{ij}{}_0 \Pi_{ij}{}^\alpha). \tag{5.75}$$

This surface integral is analogous to the surface integral that appears in the ADM Hamiltonian formulation of General Relativity (cf. Wald (1984)).

For the choice of Lagrangian (5.11) considered here, this surface integral will fall off exponentially to zero at infinity. Equation (5.74) is the canonical Hamiltonian. The total (not the canonical) Hamiltonian is

$$\begin{aligned}
H = \int d^3x \{ N J \Big[\frac{1}{J} \Pi_k{}^{\bar{l}} T^k{}_{\bar{l}\perp} + \frac{1}{2J} \Pi_{ij}{}^{\bar{l}} R^{ij}{}_{\bar{l}\perp} - \bar{\mathcal{L}}^G \Big] \\
- N n^k \nabla_\alpha \Pi_k{}^\alpha + N^\alpha \Pi_k{}^\beta T^k{}_{\beta\alpha} - N^\alpha b^k{}_\alpha \nabla_\beta \Pi_k{}^\beta + \frac{N^\alpha}{2} \Pi_{ij}{}^\beta R^{ij}{}_{\beta\alpha} \\
- A^{ij}{}_0 \Pi_{[i}{}^\alpha b_{j]\alpha} - \frac{1}{2} A^{ij}{}_0 \nabla_\alpha \Pi_{ij}{}^\alpha \} \\
+ \int d^3x \{ \Pi_k{}^0 b^k{}_{0,0} + \frac{1}{2} \Pi_{ij}{}^0 A^{ij}{}_{0,0} \} + \int d\Gamma f \varepsilon \\
- \frac{1}{2} \int d\Gamma A^{ij}{}_0 f L_{ij},
\end{aligned} \tag{5.76}$$

where the surface integral (5.75) has been dropped (notice the presence of terms relating to the momentum constraints (5.15a) and (5.15b)). A first variation of the total Hamiltonian (5.76) yields the result

$$\begin{aligned}
\delta^{(1)} H = \int d^3x \{ \frac{\delta H}{\delta N} \delta^{(1)} N + \frac{\delta H}{\delta N^k} \delta^{(1)} N^k + \frac{\delta H}{\delta b^k{}_\alpha} \delta^{(1)} b^k{}_\alpha + \frac{\delta H}{\delta \Pi_k{}^\alpha} \delta^{(1)} \Pi_k{}^\alpha \\
+ \frac{\delta H}{\delta A^{ij}{}_\alpha} \delta^{(1)} A^{ij}{}_\alpha + \frac{\delta H}{\delta \Pi_{ij}{}^\alpha} \delta^{(1)} \Pi_{ij}{}^\alpha + \frac{\delta H}{\delta A^{ij}{}_0} \delta^{(1)} A^{ij}{}_0 \\
+ \delta^{(1)} \Pi_k{}^0 b^k{}_{0,0} + \delta^{(1)} (b^k{}_{0,0}) \Pi_k{}^0 + \frac{1}{2} \delta^{(1)} \Pi_{ij}{}^0 (A^{ij}{}_{0,0}) + \frac{1}{2} \Pi_{ij}{}^0 \delta^{(1)} (A^{ij}{}_{0,0}) \} \\
+ \int d\Gamma \delta^{(1)} f \varepsilon - \frac{1}{2} \int d\Gamma \delta^{(1)} f L_{ij} A^{ij}{}_0.
\end{aligned} \tag{5.77}$$

In equation (5.77), we have the initial constraint equations (5.72), viz. $\delta H/\delta N$, $\delta H/\delta N^k$ and $\delta H/\delta A^{ij}{}_0$. These constraints may be dropped out of equation (5.77). The dynamical equations (5.38)

$$\partial_t \Pi_k{}^\alpha = -16\pi \frac{\delta H}{\delta b^k{}_\alpha},$$

$$\partial_t b^k{}_\alpha = 16\pi \frac{\delta H}{\delta \Pi_k{}^\alpha},$$

$$\partial_t A^{ij}{}_\alpha = 32\pi \frac{\delta H}{\delta \Pi_{ij}{}^\alpha}$$

and

$$\partial_t \Pi_{ij}{}^\alpha = -32\pi \frac{\delta H}{\delta A^{ij}{}_\alpha},$$

also will vanish because we are dealing with an isolated matter configuration in static equilibrium. Utilizing the primary constraints (5.15) allows one to reduce $\delta^{(1)} H$ still further. In Nikolić (1984) the multipliers $A^{ij}{}_{0,0}$ and $b^k{}_{0,0}$ are arbitrary functions of time. Perturbation of the (enforced) primary constraints (5.15) leads to the following expressions

$$\delta^{(1)} \Pi_k{}^0 = 0, \tag{5.78a}$$

$$\delta^{(1)} \Pi_{ij}{}^0 = 0. \tag{5.78b}$$

Therefore, the variation reduces to two terms

$$\delta^{(1)} H = \int d\Gamma \delta^{(1)} f \varepsilon - \frac{1}{2} \int d\Gamma \delta^{(1)} f L_{ij} A^{ij}{}_0. \tag{5.79}$$

In analogy with electromagnetic theory (cf. chapter 2), one may impose the gauge condition $A^{ij}{}_0 \equiv 0$. Finally, we have the following result

$$\int d\Gamma \varepsilon \delta^{(1)} f = \int d\Gamma \varepsilon \{g, f_o\} = -\int d\Gamma g \{\varepsilon, f_o\} = 0, \tag{5.80}$$

because the equilibrium configuration is static. We may conclude that the total Hamiltonian has a vanishing first variation

$$\delta^{(1)} H = 0. \tag{5.81}$$

5.4 Second Hamiltonian Variation $\delta^{(2)}H$

In analogy with the General Relativity and Brans-Dicke cases, one may let Y^I denote the set of variables

$$\{b^k{}_\alpha, \Pi_k{}^\alpha, A^{ij}{}_\alpha, \Pi_{ij}{}^\alpha, N, N^k, f\}. \tag{5.82}$$

The second variation of the Hamiltonian, $\delta^{(2)}H$, can be written abstractly as

$$\delta^{(2)}H = \sum_I \frac{\delta H}{\delta Y^I}\delta^{(2)}Y^I + \frac{1}{2}\sum_I\sum_J \frac{\delta^{(2)}H}{\delta Y^I \delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J, \tag{5.83}$$

where the functional derivatives are evaluated for the unperturbed equilibrium. The first term of equation (5.83) has only one non-vanishing contribution

$$\sum_I \frac{\delta H}{\delta Y^I}\delta^{(2)}Y^I = \int d\Gamma E_o \delta^{(2)} f, \tag{5.84}$$

where

$$E = \varepsilon - \frac{1}{2}A^{ij}{}_0 L_{ij}.$$

Therefore, one may write equation (5.83) as

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma \{g, E_o\}\{g, f_o\} + \frac{1}{2}\sum_I\sum_J \int d\Gamma \frac{\delta^{(2)}(fE)}{\delta Y^I \delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J$$
$$+\frac{1}{2}\sum_I\sum_J \frac{\delta^{(2)}H_G}{\delta Y^I \delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J, \tag{5.85}$$

where the second order phase-preserving perturbation $\delta^{(2)}f = \frac{1}{2}\{g, \{g, f_o\}\}$, which appears in the perturbation expansion (Bartholomew (1971))

$$f = f_o + \delta f = f_o + \{g, f_o\} + \frac{1}{2!}\{g, \{g, f_o\}\} + \frac{1}{3!}\{g, \{g, \{g, f_o\}\}\} + ..., \tag{5.86}$$

has been used. Following the calculation given in Kandrup and O'Neill (1994), we deduce

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma \{g, E_o\}\{g, f_o\} + \sum_I \int d\Gamma \delta^{(1)}Y^I \delta^{(1)}\left(f\frac{\partial E}{\partial Y^I}\right)$$

$$-\frac{1}{2}\sum_I\sum_J\int d\Gamma f\frac{\partial^2 E}{\partial Y^I\partial Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J+\frac{1}{2}\sum_I\sum_J\frac{\delta^{(2)}H_G}{\delta Y^I\delta Y^J}\delta^{(1)}Y^I\delta^{(1)}Y^J. \tag{5.87}$$

The formulae for $\partial E/\partial N$, $\partial E/\partial N^k$ and $\partial E/\partial b^l{}_\alpha$ can be used to calculate the following second term of equation (5.87)

$$\sum_I\int d\Gamma\delta^{(1)}Y^I\delta^{(1)}\left(f\frac{\partial E}{\partial Y^I}\right). \tag{5.88}$$

We have the following $\partial E/\partial Y^I$ formulae:

$$\frac{\partial E}{\partial N}=\frac{\left(E+N^\alpha(p_\alpha+\frac{1}{2}A^{kl}{}_\alpha L_{kl})\right)^2}{N^3g^{t\mu}(p_\mu+\frac{1}{2}A^{kl}{}_\mu L_{kl})}, \tag{5.89a}$$

$$\frac{\partial E}{\partial N^k}=-P_{\bar{k}}=-h_{\bar{k}}{}^\alpha P_\alpha, \tag{5.89b}$$

$$\frac{\partial E}{\partial b^k{}_\nu}=-\frac{h_k{}^\rho(p_\rho+\frac{1}{2}A^{kl}{}_\rho L_{kl})g^{\nu\mu}(p_\mu+\frac{1}{2}A^{kl}{}_\mu L_{kl})}{g^{t\mu}(p_\mu+\frac{1}{2}A^{kl}{}_\mu L_{kl})} \tag{5.89c}$$

and

$$\frac{\partial E}{\partial A^{mn}{}_\alpha}=+\frac{1}{2}\frac{g^{\alpha\mu}(p_\mu+\frac{1}{2}A^{ij}{}_\mu L_{ij})L_{mn}}{g^{t\nu}(p_\nu+\frac{1}{2}A^{kl}{}_\nu L_{kl})}. \tag{5.89d}$$

This yields the form of the second term of equation (5.87)

$$\sum_I\int d\Gamma\delta^{(1)}Y^I\delta^{(1)}\left(f\frac{\partial E}{\partial Y^I}\right)$$

$$=\int d^3x[\delta\rho\delta N+\delta J_k\delta N^k+\delta(NS_l{}^\alpha)\delta b^l{}_\alpha-\frac{1}{2}\delta A^{mn}{}_\alpha\delta\Sigma^\alpha{}_{mn}], \tag{5.90}$$

where, taking into account the fact that

$$T_{\mu\nu}=\int\frac{d^4p}{NJ}(p_\mu+\frac{1}{2}A^{ij}{}_\mu L_{ij})(p_\nu+\frac{1}{2}A^{kl}{}_\nu L_{kl})\frac{f}{m}, \tag{5.91}$$

we have:

$$\rho=JT_{\mu\nu}n^\mu n^\nu,\quad J_k=Jh_{\bar{k}}{}^\alpha T_{\alpha\mu}n^\mu$$

and

$$S_l{}^\nu=Jh_l{}^\mu T_\mu{}^\nu.$$

Also, the expression for the angular momentum tensor has been employed

$$\Sigma^{\alpha}{}_{mn} = -NJ \int \frac{d^4p}{NJ} g^{\alpha\mu}(p_\mu + \frac{1}{2}A^{kl}{}_\mu L_{kl}) L_{mn} \frac{f}{m}. \tag{5.92}$$

One writes the first order variations $\delta\rho$, δJ_k and $\delta S_l{}^\beta$ in terms of a perturbed distribution function $\delta^{(1)} f = \{g, f_o\}$. It is straightforward to calculate the third term of equation (5.87).

One must compute the following expressions

$$\frac{\partial^2 E}{\partial N^2} = \frac{\partial^2 E}{\partial N \partial N^k} = \frac{\partial^2 E}{\partial N^k \partial N^l} = 0, \tag{5.93a}$$

$$\frac{\partial^2 E}{\partial b^l{}_\alpha \partial N^k} = P_l h_{\bar{k}}{}^\alpha, \tag{5.93b}$$

as well as

$$\frac{\partial^2 E}{\partial b^l{}_\alpha \partial N} = -\frac{P_l}{NP^t}(P^t N^\alpha + P^\alpha) \tag{5.93c}$$

and

$$\begin{aligned}\frac{\partial^2 E}{\partial b^k{}_\alpha \partial b^l{}_\beta} &= h_k{}^\beta \frac{P_l P^\alpha}{P^t} + h_l{}^\alpha \frac{P_k P^\beta}{P^t} + g^{\alpha\beta}\frac{P_k P_l}{P^t} \\ &\quad - g^{t\alpha}\frac{P_k P_l P^\beta}{(P^t)^2} - g^{t\beta}\frac{P_k P_l P^\alpha}{(P^t)^2} \\ &\quad - h_k{}^t \frac{P_l P^\beta P^\alpha}{(P^t)^2} - h_l{}^t \frac{P_k P^\alpha P^\beta}{(P^t)^2} + g^{tt}\frac{P_l P^\beta P_k P^\alpha}{(P^t)^3},\end{aligned} \tag{5.93d}$$

where

$$P^\mu = g^{\mu\nu}(p_\nu + \frac{1}{2}A^{ij}{}_\nu L_{ij}) = g^{\mu\nu}P_\nu \tag{5.94}$$

and

$$P_l = h_l{}^\mu(p_\mu + \frac{1}{2}A^{ij}{}_\mu L_{ij}). \tag{5.95}$$

In addition, these expressions must also be computed

$$\frac{\partial^2 E}{\partial A^{mn}{}_\gamma \partial N} = \frac{1}{2N}\frac{L_{mn}}{P^t}\{N^\gamma P^t + P^\gamma\}, \tag{5.96a}$$

$$\frac{\partial^2 E}{\partial A^{mn}{}_\gamma \partial N^k} = -\frac{1}{2}h_{\bar{k}}{}^\gamma L_{mn}, \tag{5.96b}$$

$$\frac{\partial^2 E}{\partial A^{mn}{}_\beta \partial A^{kl}{}_\alpha} = -\frac{1}{4N^2}\frac{P^\alpha L_{kl} P^\beta L_{mn}}{(P^t)^2 P^t} - \frac{N^\alpha}{4N^2} L_{kl} \frac{P^\beta}{(P^t)^2} L_{mn}$$
$$-\frac{N^\beta}{4N^2}\frac{P^\alpha}{(P^t)^2} L_{kl} L_{mn} + \frac{1}{4}\frac{g^{\beta\alpha}}{P^t} L_{kl} L_{mn} \tag{5.96c}$$

and

$$\frac{\partial^2 E}{\partial b^k{}_\alpha \partial A^{mn}{}_\beta} = \frac{1}{2} h_k{}^t \frac{P^\beta P^\alpha}{(P^t)^2} L_{mn} - \frac{1}{2} h_k{}^\beta \frac{P^\alpha}{P^t} L_{mn}$$
$$+\frac{1}{2} h_k{}^\mu \frac{P_\mu}{(P^t)^2} g^{\alpha t} P^\beta L_{mn} - \frac{1}{2} h_k{}^\mu \frac{P_\mu}{P^t} g^{\alpha\beta} L_{mn}$$
$$-\frac{1}{2} h_k{}^\mu \frac{P_\mu P^\alpha}{(P^t)^3} P^\beta g^{tt} L_{mn} + \frac{1}{2} h_k{}^\mu \frac{P_\mu P^\alpha}{(P^t)^2} g^{t\beta} L_{mn}. \tag{5.96d}$$

Use of equations (5.93b)-(5.93d) leads to the corresponding terms

$$-\int d\Gamma f \frac{\partial^2 E}{\partial b^l{}_\alpha \partial N^k} \delta b^l{}_\alpha \delta N^k = \int d^3x N h_{\bar{k}}{}^\alpha \delta N^k \delta b^l{}_\alpha S_l{}^t, \tag{5.97a}$$

$$-\int d\Gamma f \frac{\partial^2 E}{\partial b^l{}_\alpha \partial N} \delta b^l{}_\alpha \delta N = -\int d^3x \delta N \delta b^l{}_\alpha (N^\alpha S_l{}^t + S_l{}^\alpha) \tag{5.97b}$$

and

$$-\int d\Gamma f \frac{\partial^2 E}{\partial b^k{}_\alpha \partial b^l{}_\beta} \delta^1 b^k{}_\alpha \delta^1 b^l{}_\beta$$
$$= -\int d^3x \delta^{(1)} b^k{}_\alpha \delta^{(1)} b^l{}_\beta \{-h_l{}^\alpha N S_k{}^\beta - h_k{}^\beta N S_l{}^\alpha - \frac{1}{2} g^{\beta\alpha} N S_{lk}{}^{tt} + n_l S_k{}^{\beta\alpha} + n_k S_l{}^{\beta\alpha}$$
$$+\frac{N^\alpha}{N} S_{lk}{}^{\beta t} + \frac{N^\beta}{N} S_{lk}{}^{\alpha t} + \frac{1}{2N} S_{lk}{}^{\beta\alpha}\}, \tag{5.97c}$$

where

$$S_l{}^\beta = J h_l{}^\mu T_\mu{}^\beta = J \int \frac{d^4p}{NJ} P_l P^\beta \frac{f}{m}, \tag{5.98a}$$

$$S_l{}^{\beta\alpha} = J \int \frac{d^3p dp_t}{NJ} \frac{P_l P^\beta P^\alpha}{P^t} \frac{f}{m} \tag{5.98b}$$

and

$$S_{lk}{}^{\mu\nu} = J \int \frac{d^4p}{NJ} \frac{f}{m} \frac{P_l P_k P^\mu P^\nu}{(P^t)^2}. \tag{5.98c}$$

Using equations (5.96a)-(5.96d) leads to the additional terms

$$-\int d\Gamma \frac{\partial^2 E}{\partial A^{mn}{}_\alpha \partial N} \delta A^{mn}{}_\alpha \delta N = -\frac{1}{2}\int d^3x \Sigma^\gamma{}_{mn} \delta A^{mn}{}_\gamma \frac{\delta N}{N} - \frac{1}{2}\int d^3x \frac{N^\gamma}{N} \Sigma^t{}_{mn} \delta A^{mn}{}_\gamma \delta N, \tag{5.99a}$$

$$-\int d\Gamma f \frac{\partial^2 E}{\partial A^{mn}{}_\alpha \partial N^k} \delta A^{mn}{}_\alpha \delta N^k = \frac{1}{2}\int d^3x \Sigma^t{}_{mn} \delta A^{mn}{}_\alpha h_{\bar{k}}{}^\alpha \delta N^k, \tag{5.99b}$$

$$-\int d\Gamma f \frac{\partial^2 E}{\partial A^{mn}{}_\beta \partial A^{kl}{}_\alpha} \delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha = -\frac{1}{4}\int d^3x \frac{1}{N}\Sigma^{\alpha\beta}{}_{klmn} \delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha$$

$$-\frac{1}{4}\int d^3x \frac{N^\alpha}{N}\Sigma^{t\beta}{}_{klmn} \delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha$$

$$-\frac{1}{4}\int d^3 \frac{N^\beta}{N}\Sigma^{t\alpha}{}_{klmn} \delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha$$

$$+\frac{1}{4}\int d^3x g^{\beta\alpha} N \Sigma^{tt}{}_{klmn} \delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha \tag{5.99c}$$

and

$$-\int d\Gamma f \frac{\partial^2 E}{\partial b^k{}_\alpha \partial A^{mn}{}_\beta} \delta b^k{}_\alpha \delta A^{mn}{}_\beta$$

$$= \frac{1}{2}\int d^3x \{n_k J^{\beta\alpha}{}_{mn} - h_k{}^\beta N J^{t\alpha}{}_{mn}\} \delta b^k{}_\alpha \delta A^{mn}{}_\beta$$

$$+\frac{1}{2}\int d^3x h_k{}^\mu \{\frac{N^\alpha}{N} Q_\mu{}^{t\beta}{}_{mn} - g^{\alpha\beta} N Q_\mu{}^{tt}{}_{mn} + \frac{1}{N} Q_\mu{}^{\alpha\beta}{}_{mn} + \frac{N^\beta}{N} Q_\mu{}^{t\alpha}{}_{mn}\} \delta b^k{}_\alpha \delta A^{mn}{}_\beta, \tag{5.99d}$$

where

$$\Sigma^\mu{}_{mn} = -NJ \int \frac{d^4p}{NJ} g^{\mu\nu}(p_\nu + \frac{1}{2}A^{kl}{}_\nu L_{kl}) L_{mn} \frac{f}{m}, \tag{5.100a}$$

$$\Sigma^{\mu\nu}{}_{klmn} = J \int \frac{d^4p}{NJ} \frac{P^\mu L_{kl} P^\nu L_{mn}}{(P^t)^2} \frac{f}{m}, \tag{5.100b}$$

$$J^{\mu\nu}{}_{mn} = J \int \frac{d^4p}{NJ} \frac{P^\mu P^\nu}{(P^t)} L_{mn} \frac{f}{m} \tag{5.100c}$$

and

$$Q_\mu{}^{\nu\rho}{}_{mn} = J \int \frac{d^4p}{NJ} \frac{P_\mu P^\nu P^\rho}{(P^t)^2} L_{mn} \frac{f}{m}. \tag{5.100d}$$

Using equations (5.90), (5.97a)-(5.97c) and (5.99a)-(5.99d) allows one to write the second and third terms of equation (5.87) as

$$\int d^3x \{\delta\rho\delta N + \delta J_k \delta N^k - \frac{1}{2}\delta A^{mn}{}_\alpha \delta(\Sigma^\alpha{}_{mn}) + \delta(N S_l{}^\alpha) \delta b^l{}_\alpha$$

$$+ N h_{\bar{k}}{}^\alpha \delta N^k \delta b^l{}_\alpha S_l{}^t - \delta N \delta b^l{}_\alpha (N^\alpha S_l{}^t + S_l{}^\alpha)$$

$$-\delta b^k{}_\alpha \delta b^l{}_\beta \{n_k S_l{}^{\beta\alpha} + \frac{N^\alpha}{N} S_{lk}{}^{\beta t} + \frac{1}{2N} S_{lk}{}^{\beta\alpha} - h_l{}^\alpha N S_k{}^\beta - \frac{1}{2} g^{\beta\alpha} N S_{lk}{}^{tt}\}$$

$$+\frac{1}{2}\Sigma^{t}{}_{mn}\delta A^{mn}{}_{\gamma}\delta N^{k}h_{\bar{k}}{}^{\gamma}-\frac{1}{2}\frac{\delta N}{N}\Sigma^{\gamma}{}_{mn}\delta A^{mn}{}_{\gamma}-\frac{1}{2}\frac{N^{\gamma}}{N}\Sigma^{t}{}_{mn}\delta A^{mn}{}_{\gamma}\delta N$$

$$-\frac{1}{8N}\Sigma^{\alpha\beta}{}_{klmn}\delta A^{mn}{}_{\beta}\delta A^{kl}{}_{\alpha}-\frac{N^{\alpha}}{4N}\Sigma^{t\beta}{}_{klmn}\delta A^{mn}{}_{\beta}\delta A^{kl}{}_{\alpha}$$

$$+\frac{1}{8}g^{\beta\alpha}N\Sigma^{tt}{}_{klmn}\delta A^{mn}{}_{\beta}\delta A^{kl}{}_{\alpha}$$

$$+\frac{1}{2}\{n_{k}J^{\beta\alpha}{}_{mn}-h_{k}{}^{\beta}NJ^{t\alpha}{}_{mn}\}\delta b^{k}{}_{\alpha}\delta A^{mn}{}_{\beta}$$

$$+\frac{1}{2}h_{k}{}^{\mu}\{\frac{N^{\alpha}}{N}Q_{\mu}{}^{t\beta}{}_{mn}+\frac{N^{\beta}}{N}Q_{\mu}{}^{t\alpha}{}_{mn}+\frac{1}{N}Q_{\mu}{}^{\alpha\beta}{}_{mn}-g^{\alpha\beta}NQ_{\mu}{}^{tt}{}_{mn}\}\delta b^{k}{}_{\alpha}\delta A^{mn}{}_{\beta}\}. \quad (5.101)$$

This allows us to write equation (5.87) as

$$\delta^{(2)}H=-\frac{1}{2}\int d\Gamma\{g,E_{o}\}\{g,f_{o}\}+\int d^{3}x\{\delta\rho\delta N+\delta J_{k}\delta N^{k}-\frac{1}{2}\delta A^{mn}{}_{\alpha}\delta\Sigma^{\alpha}{}_{mn}+\delta(NS_{l}{}^{\alpha})\delta b^{l}{}_{\alpha}$$

$$+Nh_{\bar{k}}{}^{\alpha}\delta N^{k}\delta b^{l}{}_{\alpha}S_{l}{}^{t}-\delta N\delta b^{l}{}_{\alpha}(N^{\alpha}S_{l}{}^{t}+S_{l}{}^{\alpha})$$

$$-\delta b^{k}{}_{\alpha}\delta b^{l}{}_{\beta}\{n_{k}S_{l}{}^{\beta\alpha}+\frac{N^{\alpha}}{N}S_{lk}{}^{\beta t}+\frac{1}{2N}S_{lk}{}^{\beta\alpha}-h_{l}{}^{\alpha}NS_{k}{}^{\beta}-\frac{1}{2}g^{\beta\alpha}NS_{lk}{}^{tt}\}$$

$$-\frac{N^{\gamma}}{2N}\Sigma^{t}{}_{mn}\delta A^{mn}{}_{\gamma}\delta N+\frac{1}{2}\Sigma^{t}{}_{mn}\delta A^{mn}{}_{\gamma}\delta N^{k}h_{\bar{k}}{}^{\gamma}-\frac{1}{2}\frac{\delta N}{N}\delta A^{mn}{}_{\gamma}\Sigma^{\gamma}{}_{mn}$$

$$-\frac{1}{8N}\Sigma^{\alpha\beta}{}_{klmn}\delta A^{mn}{}_{\beta}\delta A^{kl}{}_{\alpha}-\frac{N^{\alpha}}{4N}\Sigma^{t\beta}{}_{klmn}\delta A^{mn}{}_{\beta}\delta A^{kl}{}_{\alpha}$$

$$+\frac{1}{8}g^{\beta\alpha}N\Sigma^{tt}{}_{klmn}\delta A^{mn}{}_{\beta}\delta A^{kl}{}_{\alpha}$$

$$+\frac{1}{2}\{n_{k}J^{\beta\alpha}{}_{mn}-h_{k}{}^{\beta}NJ^{t\alpha}{}_{mn}\}\delta b^{k}{}_{\alpha}\delta A^{mn}{}_{\beta}$$

$$+\frac{1}{2}h_{k}{}^{\mu}\{\frac{N^{\alpha}}{N}Q_{\mu}{}^{t\beta}{}_{mn}-\frac{N^{\beta}}{N}Q_{\mu}{}^{t\alpha}{}_{mn}+\frac{1}{N}Q_{\mu}{}^{\alpha\beta}{}_{mn}+g^{\alpha\beta}NQ_{\mu}{}^{tt}{}_{mn}\}\delta b^{k}{}_{\alpha}\delta A^{mn}{}_{\beta}\}$$

$$+\delta^{(2)}H_{G}[\delta^{(1)}Y], \quad (5.102a)$$

where

$$\delta^{(2)}H_{G}[\delta^{(1)}Y]=+\frac{1}{2}\sum_{I}\sum_{J}\frac{\delta^{(2)}H_{G}}{\delta Y^{I}\delta Y^{J}}\delta^{(1)}Y^{I}\delta^{(1)}Y^{J}, \quad (5.102b)$$

represents the second order energy corresponding to the first order perturbation $\delta^{(1)}Y$. A standard calculation of the gravitational contribution then yields the result

$$16\pi\delta^{(2)}H_{G}[\delta^{(1)}Y]=$$

$$\int d^3x\{2\delta N\{-\frac{\delta J}{2}R^{mn}{}_{\bar{m}\bar{n}} - \frac{J}{2}h_{\bar{m}}{}^{\alpha}h_{\bar{n}}{}^{\beta}\delta R^{mn}{}_{\alpha\beta} - J\delta h_{\bar{n}}{}^{\alpha}R^{nm}{}_{\alpha\bar{m}}\}$$

$$+2\delta N^k\{\delta JR^{l\perp}{}_{\bar{l}\bar{k}} + J\delta h_{\bar{k}}{}^{\alpha}R^{l\perp}{}_{\bar{l}\alpha} + J\delta h_{\bar{l}}{}^{\beta}R^{l\perp}{}_{\beta\bar{k}} + Jh_{\bar{k}}{}^{\alpha}h_{\bar{l}}{}^{\beta}\delta R^{li}{}_{\beta\alpha}n_i\}$$

$$-\delta A^{ij}{}_0\{\delta\Pi_i{}^{\alpha}b_{j\alpha} + \Pi_i{}^{\alpha}\delta b_{j\alpha} - \delta\Pi_j{}^{\alpha}b_{i\alpha} - \Pi_j{}^{\alpha}\delta b_{i\alpha}$$

$$+\delta\Pi_{ij}{}^{\alpha}{}_{,\alpha} - \delta A^k{}_{i\alpha}\Pi_{kj}{}^{\alpha} - A^k{}_{i\alpha}\delta\Pi_{kj}{}^{\alpha}$$

$$-\delta A_j{}^k{}_{\alpha}\Pi_{ik}{}^{\alpha} - A_j{}^k{}_{\alpha}\delta\Pi_{ik}{}^{\alpha}\}\}$$

$$+\int d^3x\{N\delta^{(2)}\mathcal{H}^G{}_{\perp} + N^k\delta^{(2)}\mathcal{H}^G{}_k - \frac{1}{2}A^{ij}{}_0\delta^{(2)}\mathcal{H}^G{}_{ij}\}, \qquad (5.103)$$

where $\delta^{(2)}\mathcal{H}^G{}_{\perp}$, $\delta^{(2)}\mathcal{H}^G{}_k$ and $\delta^{(2)}\mathcal{H}^G{}_{ij}$ take the following forms:

$$\delta^{(2)}\mathcal{H}^G{}_{\perp} = -\frac{\delta^{(2)}J}{2}R^{mn}{}_{\bar{m}\bar{n}} - 2\delta J\delta h_{\bar{m}}{}^{\alpha}R^{mn}{}_{\alpha\bar{n}} - \delta Jh_{\bar{m}}{}^{\alpha}h_{\bar{n}}{}^{\beta}\delta R^{mn}{}_{\alpha\beta}$$

$$-2J\delta h_{\bar{m}}{}^{\alpha}h_{\bar{n}}{}^{\beta}\delta R^{mn}{}_{\alpha\beta} - \frac{J}{2}h_{\bar{m}}{}^{\alpha}h_{\bar{n}}{}^{\beta}\delta^{(2)}R^{mn}{}_{\alpha\beta}$$

$$-J\delta^{(2)}h_{\bar{n}}{}^{\alpha}R^{nm}{}_{\alpha\bar{m}} - J\delta h_{\bar{n}}{}^{\alpha}\delta h_{\bar{m}}{}^{\beta}R^{nm}{}_{\alpha\beta} \qquad (5.104a)$$

and

$$\delta^{(2)}\mathcal{H}^G{}_k = \delta^{(2)}JR^{l\perp}{}_{\bar{l}\bar{k}} + 2\delta J\delta R^{l\perp}{}_{\alpha\beta}h_{\bar{l}}{}^{\alpha}h_{\bar{k}}{}^{\beta}$$

$$+4\delta JR^{l\perp}{}_{\alpha\beta}h_{\bar{l}}{}^{\alpha}\delta h_{\bar{k}}{}^{\beta} + J\delta^{(2)}h_{\bar{k}}{}^{\alpha}R^{l\perp}{}_{\bar{l}\alpha}$$

$$+2J\delta h_{\bar{k}}{}^{\alpha}h_{\bar{l}}{}^{\beta}\delta R^{l\perp}{}_{\beta\alpha} + 2J\delta h_{\bar{l}}{}^{\beta}\delta h_{\bar{k}}{}^{\alpha}R^{l\perp}{}_{\beta\alpha}$$

$$+J\delta^{(2)}h_{\bar{l}}{}^{\beta}R^{l\perp}{}_{\beta\bar{k}} + 2Jh_{\bar{k}}{}^{\alpha}\delta h_{\bar{l}}{}^{\beta}\delta R^{l\perp}{}_{\beta\alpha} + Jh_{\bar{k}}{}^{\alpha}h_{\bar{l}}{}^{\beta}\delta^{(2)}R^{l\perp}{}_{\beta\alpha} \qquad (5.104b)$$

and

$$\delta^{(2)}\mathcal{H}^G{}_{ij} = \delta^{(2)}\Pi_i{}^{\alpha}b_{j\alpha} + 2\delta\Pi_i{}^{\alpha}\delta b_{j\alpha} + \Pi_i{}^{\alpha}\delta^{(2)}b_{j\alpha}$$

$$-\delta^{(2)}\Pi_j{}^{\alpha} - 2\delta\Pi_j{}^{\alpha}\delta b_{j\alpha} - \Pi_j{}^{\alpha}\delta^{(2)}b_{i\alpha}$$

$$+\delta^{(2)}\Pi_{ij}{}^{\alpha}{}_{,\alpha}$$

$$+\delta^{(2)}A^k{}_{i\alpha}\Pi_{kj}{}^\alpha - 2\delta A^k{}_{i\alpha}\delta\Pi_{kj}{}^\alpha - A^k{}_{i\alpha}\delta^{(2)}\Pi_{kj}{}^\alpha$$

$$-\delta^{(2)}A^k{}_{j\alpha}\Pi_{ik}{}^\alpha - 2\delta A^k{}_{j\alpha}\delta\Pi_{ik}{}^\alpha - A^k{}_{j\alpha}\delta^{(2)}\Pi_{ik}{}^\alpha. \tag{5.104c}$$

In the above expressions the following notations have been employed:

$$\delta J = Jh_k{}^\alpha \delta b^k{}_\alpha \tag{5.105a}$$

$$\delta^{(2)}J = Jh_{\bar{m}}{}^\beta h_{\bar{k}}{}^\alpha \delta b^{\bar{m}}{}_\beta \delta b^{\bar{k}}{}_\alpha + J\delta h_{\bar{k}}{}^\alpha \delta b^{\bar{k}}{}_\alpha + Jh_{\bar{k}}{}^\alpha \delta^{(2)} b^{\bar{k}}{}_\alpha \tag{5.105b}$$

$$\delta R^{ij}{}_{\mu\nu} = \delta A^{ij}{}_{\mu,\nu} - \delta A^{ij}{}_{\nu,\mu} + \delta A^i{}_{n\nu}A^{nj}{}_\mu + A^i{}_{n\nu}\delta A^{nj}{}_\mu$$

$$-\delta A^i{}_{n\mu}A^{nj}{}_\nu - A^i{}_{n\mu}\delta A^{nj}{}_\nu \tag{5.105c}$$

$$\delta^{(2)}R^{ij}{}_{\mu\nu} = \delta^{(2)}A^{ij}{}_{\mu,\nu} - \delta^{(2)}A^{ij}{}_{\nu,\mu} + \delta^{(2)}A^i{}_{n\nu}A^{nj}{}_\mu$$

$$+2\delta A^i{}_{n\nu}\delta A^{nj}{}_\mu + A^i{}_{n\nu}\delta^{(2)}A^{nj}{}_\mu$$

$$-\delta^{(2)}A^i{}_{n\mu}A^{nj}{}_\nu - 2\delta A^i{}_{n\mu}\delta A^{nj}{}_\nu - A^i{}_{n\mu}\delta^{(2)}A^{nj}{}_\nu \tag{5.105d}$$

Following the treatment of Kandrup and O'Neill (1994), the variational calculation takes the final form

$$\delta^{(2)}H = -\frac{1}{2}\int d\Gamma\{g, E_o\}\{g, f_o\} + \int d^3x\{-\frac{1}{2}\delta A^{mn}{}_\alpha \delta\Sigma^\alpha{}_{mn} + \delta(NS_l{}^\alpha)\delta b^l{}_\alpha$$

$$+Nh_{\bar{k}}{}^\alpha \delta N^k \delta b^l{}_\alpha S_l{}^t - \delta N \delta b^l{}_\alpha (N^\alpha S_l{}^t + S_l{}^\alpha)$$

$$-\delta b^k{}_\alpha \delta b^l{}_\beta\{n_k S_l{}^{\beta\alpha} + \frac{N^\alpha}{N}S_{lk}{}^{\beta t} + \frac{1}{2N}S_{lk}{}^{\beta\alpha} - h_l{}^\alpha N S_k{}^\beta - \frac{1}{2}g^{\beta\alpha}NS_{lk}{}^{tt}\}$$

$$-\frac{N^\gamma}{2N}\Sigma^t{}_{mn}\delta A^{mn}{}_\gamma \delta N + \frac{1}{2}\Sigma^t{}_{mn}\delta A^{mn}{}_\gamma \delta N^k h_{\bar{k}}{}^\gamma - \frac{\delta N}{2N}\delta A^{mn}{}_\gamma \Sigma^\gamma{}_{mn}$$

$$-\frac{1}{8N}\Sigma^{\alpha\beta}{}_{klmn}\delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha - \frac{N^\alpha}{4N}\Sigma^{t\beta}{}_{klmn}\delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha + \frac{1}{8}g^{\beta\alpha}N\Sigma^{tt}{}_{klmn}\delta A^{mn}{}_\beta \delta A^{kl}{}_\alpha$$

$$+\frac{1}{2}\{n_k J^{\beta\alpha}{}_{mn} - h_k{}^{\beta} N J^{t\alpha}{}_{mn}\}\delta b^k{}_{\alpha}\delta A^{mn}{}_{\beta}$$

$$+\frac{1}{2}h_k{}^{\mu}\{\frac{N^{\alpha}}{N}Q_{\mu}{}^{t\beta}{}_{mn} + \frac{N^{\beta}}{N}Q_{\mu}{}^{t\alpha}{}_{mn} + \frac{1}{N}Q_{\mu}{}^{\alpha\beta}{}_{mn} - g^{\alpha\beta}NQ_{\mu}{}^{tt}{}_{mn}\}\delta b^k{}_{\alpha}\delta A^{mn}{}_{\beta}\}$$

$$+\frac{1}{16\pi}\int d^3x\{N\delta^{(2)}\mathcal{H}^G{}_{\perp} + N^k\delta^{(2)}\mathcal{H}^G{}_k - \frac{1}{2}A^{ij}{}_o\delta^{(2)}\mathcal{H}^G{}_{ij}\} \tag{5.106}$$

SUMMARY

In chapter two, we explored the Hamiltonian structures of a collisionless Newtonian cosmology and the Vlasov-Maxwell system in a curved background space-time.

In the case of the Newtonian cosmology, one found a cosymplectic structure for the collisionless Boltzmann equation in the context of an expanding Newtonian cosmology, working in the average comoving frame. It is observed that all equilibrium solutions f_o corresponding to homogeneous distributions of matter are energy extrema with respect to dynamically accessible perturbations that preserve all the constraints associated with conservation of phase, i.e., $\delta^{(1)}H \equiv 0$.

A simple expression is derived for the second Hamiltonian variation, $\delta^{(2)}H$, associated with an arbitrary dynamically accessible perturbation, and it is shown that the time derivative $d\delta^{(2)}H/dt$ is intrinsically negative. It follows that any such perturbation with $\delta^{(2)}H < 0$ necessarily triggers a linear instability. This provides a simple proof of the linear instability of all equilibria towards perturbations of characteristic wavelength longer than the Jeans length λ_J, and of the linear instability of equilibria exhibiting population inversions towards shorter wavelength perturbations as well.

The basic approach used here to extract a nontrivial stability criterion is quite general, relying only on the facts that, in the average comoving frame, the evolution appears dissipative and the equilibrium is an energy extremum. One might , therefore, expect that it may find applications to other physical problems as well. It is important to note that the dissipative effect is intrinsic to the system itself.

The Vlasov-Maxwell analysis in a fixed curved background space-time serves as an interesting extension of the well understood Hamiltonian structure of the flat space Vlasov-Maxwell system. The formalism may be of use in studying electromagnetic effects in the neighbourhood of (for example) black holes.

Chapter three gives a completely general Hamiltonian formulation of the Vlasov-Einstein system working within the context of an ADM decomposition into space plus time. This approach incorporates explicitly the radiative degrees of freedom of the gravitational field and, as such, requires no assumptions about any underlying symmetries. This formulation is effected using lie algebraic techniques, treating the distribution function f, the spatial metric h_{ab}, and the conjugate momentum Π^{ab} as dynamical variables.

Although h_{ab} and Π^{ab} form a conjugate pair of canonical variables, f is not canonical, so that there exists no immediate analog of the canonical Hamiltonian equations and no immediate Lagrangian formulation, despite the fact that, in principle, a canonical formulation must exist (at least locally).

There are at least two concrete problems of interest for which this new Hamiltonian formulation has immediate applications. The first of these involves an examination of the stability of spherically symmetric equilibria toward nonspherical perturbations. The energy functional can be used to address the stability of more general spherical equilibria than have hitherto been studied.

Another, yet more interesting, problem involves an examination of the stability of axisymmetric equilibria, a general subject which has hitherto received absolutely no attention at all. One anticipates that realistic physical equilibria will be characterized by a nonvanishing total angular momentum but, typically, such equilibria would be expected to manifest deviations from spherical symmetry as a result of rotational flattening.

A general examination of the problem of stability for axisymmetric equilibria was attempted. Unfortunately, to obtain specific information on the stability of such matter configurations, it is necessary to have quantitative information on the behavior of a given axisymmetric solution, i.e., one must have an axisymmetric solution of the Einstein field equations (corresponding exterior and interior solutions for an axisymmetric matter configuration).

Chapter four dealt with the Brans-Dicke scalar-tensor theory. This theory has two representations:(i) the variable G representation and (ii) the variable mass representation. Only the variable mass representation has an ADM formulation, whereas the variable G representation does not possess one.

The Vlasov-Brans-Dicke system does not satisfy a Birkhoff theorem. For example, a pulsating spherically symmetric matter configuration will emit scalar monopole radiation. If the system possesses a negative energy mode, then one may have a run-away instability. The amount of energy emitted is the same for both representations. Since the variable mass representation involves a statistical mechanical description of a gas of particles with masses that have space-time dependence, the Vlasov-Brans-Dicke formalism may have a wider range of applicability to variable mass Boltzmann distributions. This dissertation gives the full form of the ADM Hamiltonian for the Vlasov-Brans-Dicke system. The gravitational part was derived by Toton (1970). However, the matter part of the ADM Hamiltonian was explicitly derived by the author. This lead to a derivation of the dynamical equations in the presence of matter. Also, a calculation of the second order Hamiltonian variation was provided.

Chapter five dealt with the Vlasov-Kibble system. The ADM form of the matter Hamiltonian was identified. It was found that the angular momentum couples to the energy via

the affine connection gravitational field variable. The theory possesses a peculiar mass shell constraint analogous to electromagnetism. The boost-angular momentum tensor L_{kl} takes the place of electrical charge e in electromagnetism.

For a torsionless gravitational Lagrangian, the *vierbein* and its conjugate momentum are static. The static nature of the *vierbein* is a key element in fixing the time independence of the spherically symmetric (Schwarzschild) solution of the theory. Thus, for a torsionless Lagrangian, a Birkhoff theorem is satisfied. However, this may not be the case for a Lagrangian with torsion. The Vlasov-Kibble system has many features in common with the Vlasov-Maxwell system. For example, the Vlasov equation of this theory is analogous to that of electromagnetism.

The specific ADM variables used in variational calculations of the Hamiltonian have also been identified. The ADM Hamiltonian can be written with respect to these variables or with respect to *vierbein* variables. In either representation the results are equivalent.

Finally, it should be noted that the formalism is applicable to any generic gravitational theory that possesses an ADM formalism.

REFERENCES

Antonov, V.A., Astr. Zh. **4**, 859 (1960) [translated in Sov. Astron. AJ **4**, 859 (1961)]

Arnold, V., Mathematical Methods of Classical Mechanics (Springer-Verlag, Berlin/New York/Heidelberg, 1978)

Arnowitt, R., Deser, S., and Misner, C.W., in Gravitation: An Introduction to Current Research, edited by L. Witten (Wiley, New York, 1962)

Bao, D., Marsden, J.E., and Walton, R., Comm. Math. Phys. **99**, 319 (1985)

Bartholomew, P., Mon. Not. R. Astr. Soc. **151**, 333 (1971)

Bernstein, I., Phys. Rev. **109**, 10 (1958)

Bisnovatyi-Kogan, G.S., and Zel'dovich, Ya. B., Astr. Zh. **47**, 758 (1970) [translated in Sov. Astron. AJ **14**, 758 (1971)]

Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., and Ratiu, T.S., in 1991 American Control Conference (IEEE, Evanston, 1991)

Brans, C., and Dicke, R.H., Phys. Rev. **124**, 925 (1961)

Bruckman, W.F., and Velaźquez, E.S., G.R.G. **25**, 901 (1993)

Chapman, S. and Cowling, T.G., The Mathematical Theory of Nonuniform Gases (Cambridge University Press, Cambridge, 1939)

Dicke, R.H., Phys. Rev. **125**, 2163 (1962)

Dirac, P.A.M., Can. J. Math. **2**, 129 (1950)

Dirac, P.A.M., Recent Developments in General Relativity and Gravitation (PWN - Polish Scientific Publications, Warsaw, 1962)

Fackerell, E.D., Astrophys. J. **160**, 859 (1970)

Fackerell, E.D., Astrophys. J. **165**, 489 (1971)

Friedman, J.L., and Schutz, B.F., Astrophys. J. **222**, 281 (1978)

Gardner, C.S., Phys. Fluids **6**, 839 (1963)

Gibbons, J., Dublin Inst. for Advanced Studies, report No. DIAS-STD-80-22 (1980)

Gibbons, J., Physica D **3**, 503 (1981)

Gordeyev, G.V., J. Exptl. Theoret. Phys. U.S.S.R. **6**, 660 (1952)

Holm, D.D., Physica D **25**, 261 (1987)

Holm, D.D., Marsden, J.E., Ratiu, T.S., and Weinstein, A., Phys. Repts. **123**, 1 (1985)

Ipser, J.R., Astrophys. J. **156**, 509 (1969a)

Ipser, J.R., Astrophys. J. **158**, 17 (1969b)

Ipser, J.R., Astrophys. J. **241**, 1101 (1980)

Ipser, J.R., and Horwitz, G., Astrophys. J. **232**, 863 (1979)

Ipser, J.R., and Semenzato, R., Astrophys. J. **227**, 590 (1979)

Ipser, J.R., and Thorne, K.S., Astrophys. J. **154**, 251 (1968)

Israel, W., in General Relativity: Papers in Honour of J.L. Synge, edited by L. O'Raifeartaigh (Clarendon, Oxford, 1972)

Iwinski, Z.R., and Turski, L.A., Lett. Appl. Eng. Sci. **4**, 179 (1976)

Kandrup, H.E., Astrophys. J. **351**, 104 (1990)

Kandrup, H.E., Astrophys. J. **370**, 312 (1991a)

Kandrup, H.E., Astrophys. J. **380**, 511 (1991b)

Kandrup, H.E., and Morrison, P.J., Ann. Phys. (N.Y.) **225**, 114 (1993)

Kandrup, H.E., and O'Neill, E., Phys. Rev. D **47**, 3229 (1993a)

Kandrup, H.E., and O'Neill, E., Phys. Rev. D **48**, 4534 (1993b)

Kandrup, H.E., and O'Neill, E., Phys. Rev. D **49**, 5115 (1994)

Kandrup, H.E., and Sygnet, J.F., Astrophys. J. **298**, 27 (1985)

Kibble, T.W.B., J. Math. Phys. **2**, 212 (1961)

Kruskal, M.D., and Oberman, C., Phys. Fluids **1**, 275 (1958)

Lynden-Bell, D., and Sanitt, N., Mon. Not. R. Astr. Soc. **143**, 167 (1969)

Marsden, J.E., Montgomery, R., Morrison, P.J., and Thompson, W.B., Ann. Phys. (N.Y.) **169**, 29 (1986)

Marsden, J.E., and Weinstein, A., Physica D **4**, 394 (1982)

Misner, C.W., Thorne, K.S., and Wheeler, J.A., Gravitation (W.H. Freeman, San Francisco, 1973)

Moncrief, V., J. Math. Phys. **17**, 1893 (1976)

Morrison, P.J., Phys. Lett. A **80**, 383 (1980a)

Morrison, P.J., Phys. Rev. **45**, 790 (1980b)

Morrison, P.J., Z. Naturforsch. Teil A **42**, 1115 (1987)

Morrison, P.J., and Greene, J.M., Phys. Rev. Lett. **45**, 790 (1980)

Morrison, P.J., and Pfirsch, D., Phys. Rev. A **40**, 3898 (1989)

Morrison, P.J., and Pfirsch, D., Phys. Fluids B **2**, 1105 (1990)

Morrison, P.J., and Pfirsch, D., Phys. Fluids B **4**, 3038 (1992)

Moser, J.K., Mem. Am. Math. Soc. **81**, 1 (1968)

Newcomb, W.A., Ann. Phys. **3**, 347 (1958)

Nikolić, I.A., Phys. Rev. D **30**, 2508 (1984)

Nutku, Y., Astrophys. J. **155**, 999 (1969)

Peebles, P.J.E., The Large Scale Structure of the Universe (Princeton University Press, 1980)

Penrose, O., Phys. Fluids **3**, 258 (1963)

Reddy, D.R.K., J. Phys. A: Math., Nucl. Gen. **6**, 1867 (1973)

Shapiro, S.L., and Teukolsky, S.A., Astrophys. J. **419**, 622 (1993a)

Shapiro, S.L., and Teukolsky, S.A., Astrophys. J. **419**, 635 (1993b)

Schutz, B.F., Phys. Rev. D **2**, 2762 (1970)

Stewart, J.M., Nonequilibrium Relativistic Kinetic Theory (Springer-Verlag, Berlin, 1971)

Sygnet, J.F., Des Forets, G., Lachize-Rey, M., and Pellat, R., Astrophys. J. **246**, 737 (1984)

Toton, E.T., J. Math. Phys. **11**, 1713 (1970)

Utiyama, R., Phys. Rev. **101**, 1597 (1956)

Wagoner, R.V., Phys. Rev. D **1**, 3209 (1970)

Wald, R.M., General Relativity (University of Chicago Press, Chicago, 1984)

Yang, C.N., and Mills, R.L., Phys. Rev. **96**, 191 (1954)

BIOGRAPHICAL SKETCH

Eric O'Neill was born on November 17, 1964, in Sydney, Nova Scotia, Canada. His parents are Ronald Joseph O'Neill and Eugenie Elizabeth (neé Morrison) O'Neill. During the years 1966-67, he lived in Kingston, Ontario, and his sister, Rhonda, was born there in 1966. His family moved back to Nova Scotia in 1968. He started school in September 1970.

He became interested in science at a fairly early age. His parents gave him a telescope as a Christmas gift. A total eclipse of the sun occurred in Nova Scotia in March 1970, and this helped propel his early interest in astronomy. The Apollo Moon missions were also a source of endless fascination.

He started high school in September 1980. After difficulties with mathematics, he became interested in mathematical logic after reading a science fiction novel. Soon he started studying mathematics on his own as an interest. After finishing an honours B.Sc. at Dalhousie University, he worked with Professor Alan Coley of the mathematics department for an M.Sc.

In September 1989 he arrived at the University of Florida. He passed the physics comprehensive exams in August 1990. He then worked with Professor Henry Emil Kandrup towards the Ph.D. in physics.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Henry E. Kandrup, Chair
Professor of Physics

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May 2000

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